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Vibration of cantilevered laminated composite conical shells with twist

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Abstract

Considering an accurate relationship between strains and displacements derived on the thin shell theory for twisted thin conical shells, a method for analyzing the vibrations of twisted conical shells made up of laminated composite materials is presented, in which the principle of virtual work for free vibrations and the Rayleigh-Ritz procedure with algebraic polynomials in two elements as displacements are adopted. The effects of the number of laminae, the fiber orientation, the twist, the taper ratio and the subtended angle on the vibration characteristics of cantilevered laminated composite conical shells are studied by the present method.

1 Introduction

With the developments of new materials and relative techniques, composite materials due to their superior properties are becoming increasingly used in many engineering applications as structural elements, especially in the aerospace, turbomachinery and ship building industrials in the form of laminated composite plates and shells. The study in this paper focuses on the free vibrations of blades.

To the physical models of blades, there were a lot of models such as beams, twisted plates and shallow cylindrical shells made up of isotropic materials, although twisted conical shells are more appropriate for representing blades, there were a few researches about shallow conical shell [1]. Recently, some work on the vibrations of open, laminated composite cantilevered shallow conical shells were reported [2,3].

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Based on the study of twisted conical shells in reference [4], considering an accurate relationship between strains and displacements of twisted thin conical shells derived on the thin shell theory, a method for free vibration analysis of laminated composite conical shells with twist is proposed by the principle of virtual work and the Rayleigh-Ritz procedure with algebraic polynomials in two elements as admissible displacement functions. The accuracy and the validity of the method are verified by way of the comparisons, and the effects of the laminated constructional and geometric parameters such as the number of laminae, the fiber orientation, the twist, the taper ratio and the subtended angle on the vibration characteristics are investigated.

2 Formulation

A geometry of twisted laminated composite conical shells and a coordinate system are given in Figure 1. A thin laminated composite conical shell is considered and a lamina is assumed to be specially orthotropic and perfectly bonded to the adjacent laminae in here.

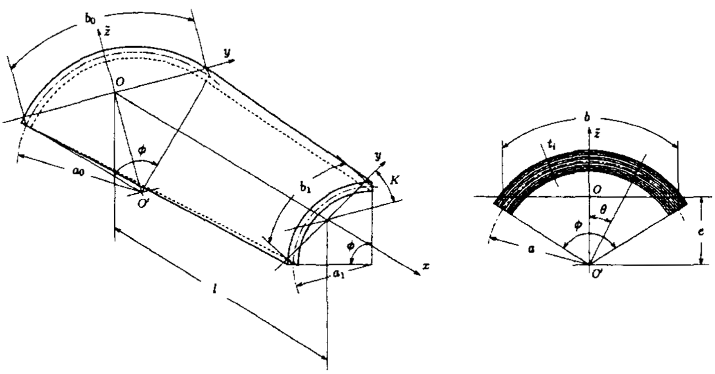

Figure 1: A geometry of a twisted laminated composite conical shell.

It is assumed that the displacement components for a point on the midsurface are u, v and w, respectively, then the displacement components U, V and W, and the displacement vector $\mathbf{\Gamma}$ for an arbitrary point in normal direction of the midsurface at a distance z from the midsurface are given as follows [4]:

$$U = u - \frac{z}{g}\left(\frac{p_{,x}}{\sqrt{g}}u + \frac{p_{,\theta}}{a\sqrt{g}}v + \frac{\partial w}{\partial x} - \frac{q}{a}\frac{\partial w}{\partial \theta}\right), \qquad W = w,$$

$$V = v - \frac{z}{\sqrt{g}}\left[\left(k - \frac{p_{,x}q}{g}\right)u - \frac{1}{a}\left(1 + \frac{p_{,\theta}q}{g}\right)v - \frac{q}{\sqrt{g}}\frac{\partial w}{\partial x} + \frac{\sqrt{g}}{a}\left(1 + \frac{q^2}{g}\right)\frac{\partial w}{\partial \theta}\right],$$

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$$\boldsymbol{\Gamma}=\left(U-\frac{p}{\sqrt{g}}W\right)\boldsymbol{i}_1+\left(fU+V\cos\theta+\frac{\sin\theta}{\sqrt{g}}W\right)\boldsymbol{i}_2+\left(hU-V\sin\theta+\frac{\cos\theta}{\sqrt{g}}W\right)\boldsymbol{i}_3, \quad (1)$$

where the prime in subscript denotes partial differentiation and the variables in the equations are defined in Appendix A. The strains can be given as

$$\begin{bmatrix}\varepsilon_{\xi\xi}\\ \varepsilon_{\eta\eta}\\ \gamma_{\xi\eta}\end{bmatrix}=\frac{1}{1-\bar{m}\frac{z}{l}}\boldsymbol{ZFU}+\frac{1}{2}\begin{bmatrix}W_\xi^2\\ W_\eta^2\\ 2W_\xi W_\eta\end{bmatrix}, \quad (2)$$

where $\boldsymbol{Z}$, $\boldsymbol{F}$, $\boldsymbol{U}$, W_ξ and W_η are

$$\boldsymbol{Z}=\begin{bmatrix}1 & \frac{z}{l} & \frac{z^2}{l^2} & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & \frac{z}{l} & \frac{z^2}{l^2} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{z}{l} & \frac{z^2}{l^2}\end{bmatrix}, \quad \boldsymbol{F}=[F_{i,j}],\left(\begin{matrix}i=1\sim 9\\ j=1\sim 12\end{matrix}\right),$$

$$\boldsymbol{U}^T=\left[\frac{\partial\bar{u}}{\partial X}\ \ \frac{\partial\bar{u}}{\partial\theta}\ \ \bar{u}\ \ \frac{\partial\bar{v}}{\partial X}\ \ \frac{\partial\bar{v}}{\partial\theta}\ \ \bar{v}\ \ \frac{\partial^2\bar{w}}{\partial X^2}\ \ \frac{\partial^2\bar{w}}{\partial\theta^2}\ \ \frac{\partial^2\bar{w}}{\partial X\partial\theta}\ \ \frac{\partial\bar{w}}{\partial X}\ \ \frac{\partial\bar{w}}{\partial\theta}\ \ \bar{w}\right],$$

$$W_\xi=\frac{\bar{p}_{,x}}{g}\bar{u}+\frac{p_{,\theta}}{\bar{a}g}\bar{v}+\frac{1}{\sqrt{g}}\frac{\partial\bar{w}}{\partial x}-\frac{q}{\bar{a}\sqrt{g}}\frac{\partial\bar{w}}{\partial\theta}, \quad W_\eta=\frac{K}{\sqrt{g}}\bar{u}-\frac{1}{\bar{a}\sqrt{g}}\bar{v}+\frac{1}{\bar{a}}\frac{\partial\bar{w}}{\partial\theta}. \quad (3)$$

For the purpose of parametric analysis, the following dimensionless quantities are utilized in this paper,

$$\bar{u}=\frac{u}{l},\ \ \bar{v}=\frac{v}{l},\ \ \bar{w}=\frac{w}{l},\ \ X=\frac{x}{l},\ \ K=kl,\ \ \bar{a}=\frac{a}{l},\ \ \bar{m}=ml,\ \ \bar{p}_{,x}=p_{,x}l, \quad (4)$$

The stress-strain relationship for a lamina is

$$\begin{bmatrix}\sigma_{\xi\xi}\\ \sigma_{\eta\eta}\\ \tau_{\xi\eta}\end{bmatrix}=\boldsymbol{Q}\begin{bmatrix}\varepsilon_{\xi\xi}\\ \varepsilon_{\eta\eta}\\ \gamma_{\xi\eta}\end{bmatrix}, \quad (5)$$

where $\boldsymbol{Q}$ is the transformed reduced stiffness matrix.

The principle of virtual work for the present problem can be written as follows:

$$\iiint_m \delta\begin{bmatrix}\varepsilon_{\xi\xi} & \varepsilon_{\eta\eta} & \gamma_{\xi\eta}\end{bmatrix}\begin{bmatrix}\sigma_{\xi\xi}\\ \sigma_{\eta\eta}\\ \tau_{\xi\eta}\end{bmatrix}\sqrt{g}\left(1-\bar{m}\frac{z}{l}\right)dxad\theta dz-\iiint_m \rho\omega^2\boldsymbol{\Gamma}\,\delta\boldsymbol{\Gamma}\sqrt{g}\left(1-\bar{m}\frac{z}{l}\right)dxad\theta dz=0, \quad (6)$$

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where ρ is a density of a material and ω is an angular frequency. Substituting Eqs. (1), (2), (4) and (5) into Eq. (6), integrating with respect to z by neglecting the terms having z^i where i is greater than 3, and multiplying l^2/E_1t^3, the dimensionless principle of virtual work for the free vibrations of twisted laminated composite conical shells is

$$\iint_S \delta\boldsymbol{U}^T\boldsymbol{F}^T\boldsymbol{D}\boldsymbol{F}\boldsymbol{U}\sqrt{g}\,\bar{a}\,dXd\theta - \lambda^2\iint_S\left[\left(g+q^2\right)\bar{u}\delta\bar{u}+q\bar{u}\delta\bar{v}\right.$$

$$\left.+q\bar{v}\delta\bar{u}+\bar{v}\delta\bar{v}+\bar{w}\delta\bar{w}\right]\sqrt{g}\,\bar{a}\,dX\,d\theta=0, \qquad (7)$$

where a dimensionless frequency parameter λ is defined as

$$\lambda=\frac{\omega\,l^2}{t}\sqrt{\frac{\rho}{E_1}}, \qquad (8)$$

E_1 is Young's modulus of a composite material in one direction, t ($t=\sum_{k=1}^{N}t_k$, N is the number of laminae) is a thickness of a conical shell and $\boldsymbol{D}$ is

$$\boldsymbol{D}=\begin{bmatrix}\boldsymbol{D}_{11} & \boldsymbol{D}_{12} & \boldsymbol{D}_{13}\\ & \boldsymbol{D}_{22} & \boldsymbol{D}_{23}\\ Sym. & & \boldsymbol{D}_{33}\end{bmatrix}, \quad \boldsymbol{D}_{ij}=\begin{bmatrix}A_{ij}+\bar{m}B_{ij} & B_{ij}+\bar{m}C_{ij} & C_{ij}\\ & C_{ij} & 0\\ Sym. & & 0\end{bmatrix}, \quad (9)$$

and A_{ij}, B_{ij} and C_{ij} denote the stretching stiffness, the membrane-bending coupling and the bending stiffness of a laminated shell.

With the demand of the Rayleigh-Ritz procedure, the displacements $\bar{u},\bar{v}$ and $\bar{w}$ are assumed as two-dimensional algebraic polynomial functions of X and θ that satisfy the geometric boundary conditions employed at an end of twisted laminated composite conical shells ($X=0$, $\bar{u}=0$, $\bar{v}=0$, $\bar{w}=0$ and $\partial\bar{w}/\partial X=0$), or

$$\bar{u}=\sum_{i=1}^{N_{\bar{u}}}\sum_{j=0}^{M_{\bar{u}}}a_{ij}X^i\theta^j, \quad \bar{v}=\sum_{k=1}^{N_{\bar{v}}}\sum_{l=0}^{M_{\bar{v}}}b_{kl}X^k\theta^l, \quad \bar{w}=\sum_{m=2}^{N_{\bar{w}}}\sum_{n=0}^{M_{\bar{w}}}c_{mn}X^m\theta^n. \quad (10)$$

Substituting Eq. (10) into Eq. (7), then taking the variation of the equation with respect to the unknown quantities a_{ij}, b_{kl} and c_{mn} according to the Rayleigh-Ritz procedure yields a set of homogeneous equations in matrices,

$$(\boldsymbol{A}-\lambda^2\boldsymbol{B})\boldsymbol{q}=0, \qquad (11)$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are the stiffness and mass matrices, respectively, and the vector $\boldsymbol{q}$ denotes the amplitude of vibrations. The characteristic equation can be obtained according to the requirement of a nontrivial solution.

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3 Discussions

Twisted conical shells made up of E-glass/epoxy composite material ($E_1 =$ 60.7 Gpa, $E_2 = 24.8$ Gpa, $G_{12} = 12.0$ Gpa, $\nu = 0.23$) are studied and the first four frequency parameters λ of vibration are given.

3.1 Convergence and validity

First, the convergence of the numerical results by the present method is checked and it is known that the numerical integral with 16-point and the displacements $\bar{u}$, $\bar{v}$ and $\bar{w}$ with 64, 64 and 72 items, or the maximum powers of X and θ are 8 and 7 in $\bar{u}$ and $\bar{v}$, and 9 and 8 in $\bar{w}$, are enough for the convergence of the frequency parameters λ, which are utilized in the following analyses.

Second, the cantilevered shallow conical shells with stacking sequence $[(-\theta_f, \theta_f)_2]_{unsym}$ [2] are considered. As indicated in Table 1, the present results agree with the previous ones well, although the differences in the second frequency parameters are greater where the maximum is 4.2%, the most are less than 1%.

3.2 Effects on vibrations

To twisted conical shells with four, eight and twelve laminae and symmetric angle-cross stacking sequences, the effect of laminae on vibrations is investigated, which are shown in Table 2. It can be seen that all the frequency parameters λ trend towards ascend with the number of laminae increasing in the case of $K = 0°$, although most of the variations are small. In the case of $K = 30°$, only the first frequency parameter λ shows monotony and which decreases with the number of laminae increasing, and the changes of the others can be seen, but they are complicated. It means that the twist makes for the coupling vibrations occurring and counteracts the effect of the laminae increasing on the vibration characteristics, and which is greater than that of the laminae. Due to the small the effect of the laminae, the variations of the λ are almost the same as the fiber orientation α varies for the laminated composite conical shells with the different laminae, but there are the differences with the α for the different types of vibration modes. When the fibers are perpendicular to the lengthwise direction, or $\alpha = 90°$, the λ always becomes the minimum for laminated composite conical shells.

For the four-lamina, symmetric stacking sequence laminated composite conical shells, the effect of the subtended angle ϕ on the frequency parameters λ is investigated, which are given in Table 3. In the both cases of $K = 0°$ and 30°, all the frequency parameters increase with the ϕ increasing, because the curvature in a chordwise direction increases with the subtended

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angle ϕ increases when the aspect ratios $(b_1/l,\ b_0/l)$ are constant and it causes the bending and torquing stiffnesses of the laminated composite conical shells to increase. Also, it can be seen that the variations with the ϕ are quite different for different vibration modes, Usually, the variation of the bending mode is greater than that of the torquing one.

Table 4 shows the effects of the twist angle and the taper ratio (b_1/b_0) on the frequency parameters of the conical shells with symmetric angle-cross stacking sequence $[(+\alpha/-\alpha)]_s$. Most of the frequency parameters have ascending tendencies with the decrease of the taper ratio, specially for the laminated composite conical shells with great twist. The complicated effect of the twist on the frequency parameters can be seen, there is mere the first λ having the monotony with the twist angle, or it decreases with the K increasing, and there are undefined tendencies for the others. Considering the fiber orientation α, no matter the other parameters of conical shells are, the λ always becomes the minimum when the fibers are perpendicular to the lengthwise direction, or $\alpha=90°$, but the maximum of the λ do not always appear in the case of $\alpha=0°$ due to the curvature and the twist of shells.

4 Conclusions

A method for analyzing the free vibrations of cantilevered laminated composite conical shells with twist is proposed, in which an accurate strain-displacement relationship on the thin shell theory is considered, the energy equilibrium equation is obtained by the principle of virtual work, and the eigenvalue equation is formulated using the Rayleigh-Ritz procedure with two dimensional polynomials as admissible displacement functions.

The effects of the laminated constructional and geometric parameters on the vibration characteristics are investigated. It is known that the effect of the subtended angle is consistent for all the frequency parameters, or they vary monotonically with it. Only the first frequency parameter shows monotony with the twist angle and there are a ascending tendency for all the frequency parameters with the decrease of the taper ratio. The increase of laminae makes all the frequency parameters of untwisted laminated composite conical shells increase, however which are small, the first frequency parameter decreases with the increase of laminae and the others are irregular under the effect of twist, which means that the effect of the twist is greater than that of the coupling terms of laminated constructions. The minimum frequency parameter always appears when the fibers are perpendicular to the lengthwise direction and the maximum one are undefined.

Table 1. Comparisons of λ ($a/h=100.0$, $a/b_0=1.5$, $\theta_v=15°$, $\theta_0=30°$, $\psi=15°$, $[(-\theta_f, \theta_f)_2]_{unsym}$)

Method	Ref. [2]				Present			
Mode	1	2	3	4	1	2	3	4
θ_f 0°	0.021271	0.066337	0.12163	0.13323	0.021357	0.063855	0.12134	0.13359
15°	0.021431	0.066187	0.12220	0.13351	0.021571	0.063526	0.12287	0.13353
30°	0.021362	0.063875	0.12016	0.13726	0.021449	0.061552	0.12171	0.13623
45°	0.020405	0.059428	0.11332	0.14583	0.020386	0.057619	0.11439	0.14449
60°	0.018954	0.055037	0.10550	0.15022	0.018885	0.053508	0.10617	0.14849
75°	0.017578	0.051632	0.098925	0.14407	0.017525	0.050150	0.099509	0.14244
90°	0.016933	0.050176	0.095991	0.14031	0.016918	0.048674	0.096758	0.13848

Table 2. Effect of laminae on λ ($b_0/t=25$, $b_0/l=0.5$, $b_1/l=0.3$, $\phi=60°$)

		$K=0°$				$K=30°$			
	α	1	2	3	4	1	2	3	4
$[(+\alpha/-\alpha)]_s$	15°	3.4420	5.5840	14.843	15.966	2.6634	7.2939	12.366	18.342
	45°	2.7045	5.9384	12.747	16.223	2.3277	6.9740	10.717	18.070
	75°	2.3680	5.1247	11.213	14.050	1.9627	5.9765	9.2545	15.440
$[(+\alpha/-\alpha)_2]_s$	15°	3.4469	5.5996	14.895	16.010	2.5884	7.2124	12.256	18.136
	45°	2.7093	6.0854	12.776	16.603	2.2793	7.0186	10.644	18.103
	75°	2.3681	5.1967	11.214	14.208	1.9583	6.0229	9.2394	15.521
$[(+\alpha/-\alpha)_3]_s$	15°	3.4472	5.6004	14.898	16.012	2.5735	7.1947	12.233	18.089
	45°	2.7096	6.0930	12.778	16.623	2.2682	7.0099	10.624	18.064
	75°	2.3681	5.2005	11.214	14.217	1.9565	6.0234	9.2345	15.517

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Table 3. Effect of ϕ on λ ($b_0/t=25$, $b_0/l=0.5$, $b_1/l=0.3$)

		$K=0°$				$K=30°$			
ϕ	α	1	2	3	4	1	2	3	4
30°	0°	2.0761	5.1981	9.6958	14.902	1.6532	6.7007	9.0321	16.797
	45°	1.5686	5.8197	7.8232	15.722	1.4328	6.3950	7.6450	16.722
	90°	1.3595	4.8140	6.7393	13.164	1.1635	5.1831	6.5296	13.509
60°	0°	3.5585	5.3635	15.024	15.516	2.5637	7.0856	12.326	17.654
	45°	2.7045	5.9384	12.747	16.223	2.3277	6.9740	10.717	18.070
	90°	2.3464	4.9075	11.030	13.522	1.9066	5.7683	9.0272	14.847
90°	0°	5.0499	5.5917	16.405	19.970	3.5423	7.4341	15.069	19.343
	45°	3.8395	6.0741	16.669	17.599	3.2614	7.0855	13.741	18.876
	90°	3.3347	5.0318	13.975	15.337	2.6954	5.9201	11.537	15.736

Table 4. Effect of K and b_1/b_0 on λ ($b_0/t=25$, $b_0/l=0.5$, $\phi=60°$, $[(+\alpha/-\alpha)]_s$)

		$b_1/b_0=0.2$				$b_1/b_0=0.6$			
K	α	1	2	3	4	1	2	3	4
0°	0°	4.1318	9.2178	14.322	22.608	3.5585	5.3635	15.024	15.516
	45°	3.1618	10.061	11.657	20.674	2.7045	5.9384	12.747	16.223
	90°	2.7407	8.3531	10.080	17.636	2.3464	4.9075	11.030	13.522
15°	0°	3.8619	9.8843	13.669	23.494	3.1327	6.0500	13.568	16.759
	45°	3.1084	10.457	11.336	21.137	2.6028	6.2941	11.974	17.207
	90°	2.6403	8.6778	9.6859	18.116	2.1960	5.2065	10.163	14.272
30°	0°	3.3525	11.100	12.692	25.027	2.5637	7.0856	12.326	17.654
	45°	2.9024	10.354	11.476	22.191	2.3277	6.9740	10.717	18.070
	90°	2.4087	8.7047	9.6119	19.058	1.9066	5.7683	9.0272	14.847
45°	0°	2.9003	10.251	14.149	24.627	2.1555	7.3163	12.338	17.940
	45°	2.6513	9.1849	12.844	22.447	2.0527	7.1320	10.356	18.164
	90°	2.1538	7.6490	10.808	19.023	1.6422	5.8162	8.7866	14.782
60°	0°	2.5632	8.8998	16.404	23.244	1.8834	6.6604	13.536	17.667
	45°	2.4179	8.0476	14.359	21.502	1.8296	6.4368	11.189	17.614
	90°	1.9302	6.6356	12.134	17.942	1.4386	5.1778	9.5503	14.177

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Appendix A

The quantities used in this paper are defined as the following,

$$f = a_{,x}\sin\theta - k(a\cos\theta - e), \quad h = a_{,x}\cos\theta - e_{,x} + ka\sin\theta,$$

$$p = f\sin\theta + h\cos\theta, \quad g = 1 + p^2, \quad q = f\cos\theta - h\sin\theta, \quad d_3 = ap_{,x} - p_{,\theta}q,$$

$$m = -\frac{1}{a\sqrt{g}}\left(1 - \frac{d_3}{g}\right). \tag{12}$$