Transactions on the Built Environment vol 57, © 2001 WIT Press, www.witpress.com, ISSN 1743-3509

Numerical simulation of structures equipped with friction energy dissipation devices subjected to seismic load

S.T. De la Cruz[1], F. Lopez-Almansa[2], L. Bozzo[3], L. Pujades[4]
[1] *Technical University of Catalonia, Geotechnical Engineering and Geosciences Dept., Spain*
[2] *Technical University of Catalonia, Architecture Structures Dept., Spain*
[3] *University of Girona, Industrial Engineering Dept., Spain.*
[4] *Technical University of Catalonia, Geotechnical Engineering and Geosciences Dept., Spain.*

Abstract

The safety and the prevention of collapsing of buildings designed for inhabitation have been two of the major concerns of humanity since the early times. For this, since the early 70's various devices have been designed to give an amount of extra damping to structures, such as: metallic, visco-elastic and frictional devices. These latter have proved to be adequate and cheap. For this and other reasons, this paper deals with the numerical simulation of structures equipped with friction energy dissipation devices. The dynamic behavior of such devices is highly non-linear, therefore at the present there is a lack of precise and reliable numerical models, and the clear demonstration of the efficiency of these elements is not completely fulfilled. A non-linear numerical model developed by the authors simulates the dynamic behavior of structures incorporating friction energy dissipation devices. This numerical model consists of an algorithm that carries out step-by-step time integration, and it's based on Newmark's method. The friction force acting on the dissipators is assumed to follow Coulomb's law of dry friction. The results obtained are compared to those got with the commercial software ADINA, and the agreement is good. Besides, preliminary results show that friction dissipators reduce substantially the dynamic response of structures. The numerical model can simulate the response of structures when affected by dynamic actions, such as wind and earthquake loads.

Transactions on the Built Environment vol 57, © 2001 WIT Press, www.witpress.com, ISSN 1743-3509

1 Introduction

The necessity of reducing the response of buildings when subjected to dynamic loads is of a very great concern for most structural engineers, both for preventing structural collapse (ultimate limit state) and for serviceability conditions (human comfort, among others). Traditional approach has consisted of dissipating the input energy through big strains in the main structure, but this causes damage and/or leads to conservative and unpractical designs. Conversely, in the early 70's the use of devices that are not a part of the main load-carrying system was suggested [1,2]. These devices are specifically designed to absorb the input energy and they can be easily replaced as soon as any strong excitation acting on the structure has finished. Figure 1*a* shows a 2D multi-story building where dissipators have been placed in the connections between the main structure and the bracing system.

To describe the dynamic behavior of buildings incorporating energy dissipators, general nonlinear models are non-suitable. In such models the main structure is supposed to remain elastic while the nonlinearities are left to dissipators. Besides, the coexistence of elements with extremely different stiffness parameters (the dissipators are generally less stiff than the main structure) can lead to numerical instability and to a certain lack of accuracy. Hence, a number of numerical models have been developed, as the one considered in the ADINA program (this includes general contact problems) [3].

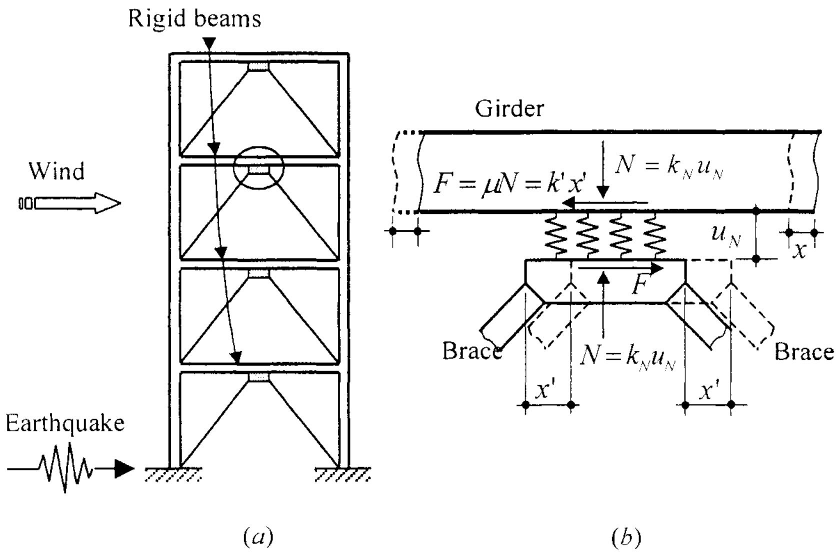

Figure 1: Structure equipped with energy dissipators: (*a*) 2D frame modelled as a shear building, (b) detail of the friction dissipator

Transactions on the Built Environment vol 57, © 2001 WIT Press, www.witpress.com, ISSN 1743-3509

1.1 Friction dampers

Among the existing energy dissipation devices, three major types are currently used: devices based on *yielding of metals*, *friction dampers* and *viscous or visco-elastic dampers* [1].

This paper focuses on the analysis of friction devices, because these dissipators offer several advantages:

- They have a high-energy dissipation capacity
- The load amplitude, the input frequency contents or the number of loading cycles does not affect significantly their behavior
- They have a good reliability and a long durability

Besides, their dynamic behavior is highly nonlinear, so the numerical simulation is a challenging issue; this causes some subjects to be controversial, e g, their high-frequency response or their seismic behavior under near-fault pulses.

There are a variety of friction devices that have been proposed for structural energy dissipation [1]. These devices differ in their mechanical complexity and in the material used in the sliding surfaces. However, if it is assumed that the friction coefficient is non-velocity dependent and the prestressing force is constant, almost all of them generate rectangular hysteresis loops typical of Coulomb friction, as those shown in Figure 2.

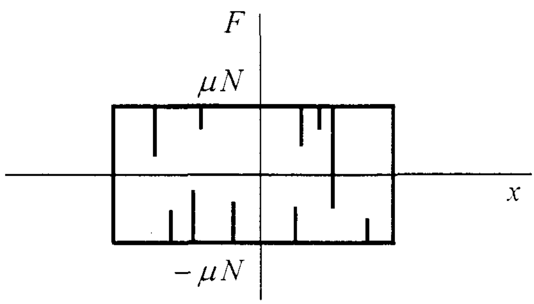

Figure 2: Dry friction hysteresis loops

2 Friction constitutive model

The static behavior of a single friction dissipator is described in this section [4]. Figure 1*b* shows the mechanical model of the contact problem.

In Figure 1*b* x and x' represent, respectively, the horizontal displacements of the main frame and the dissipation device. Coefficient k' is the stiffness of the bracing system supporting the dissipator.

In a single point belonging to the contact surface, the limit condition for the unidirectional constitutive model —based on Coulomb's law— is

$$f(F,u_N)=g(F,u_N)=|F|-|\mu N|=|F|-|\mu K_N u_N|\leq 0 \qquad (1)$$

where $f(F,u_N)$ and $g(F,u_N)$ are the plastic yielding limit function and the plastic potential, respectively. F is the friction force between the dissipator and the structure, μ is the dry friction coefficient ($\mu=\tan\phi^{fric}$ where ϕ^{fric} is the roughness angle) and N is the prestressing force —acting normally to the contact surface— given by $N=K_N u_N$, where K_N and u_N are the penetration stiffness and the displacement, respectively.

If, inside the calculation process this limit condition is not fulfilled, it means there is a sliding ($\dot{x}\neq\dot{x}'$). Moreover, if there is sliding, the maximum friction force is given by

$$F=\mu N \qquad (2)$$

as stated on Coulomb's law of dry friction [5].

3 Equations of motion for multi-story frames

In this section the building with N floors and incorporating friction dissipators, as described in Figure 1, is considered. The external excitation consists of a seismic motion; however, the case of wind loading can be similarly analyzed as shown next.

The dynamic structural behavior is modelled as a 2D shear building. The degrees of freedom are the relative horizontal displacements between the floors $(x_1, x_2, \ldots, x_N)$ and the dissipators $(x'_1, x'_2, \ldots, x'_N)$. The number of degrees of freedom ranges between N (there is not sliding at any dissipator) and $2N$ (all dissipators are sliding simultaneously).

The equations of motion of the $2N$ degrees of freedom are:

$$m_1(\ddot{x}_1+\ddot{x}_g)+c_1\dot{x}_1+k_1x_1-c_2(\dot{x}_2-\dot{x}_1)-k_2(x_2-x_1)$$
$$-c'_2(\dot{x}'_2-\dot{x}_1)-k'_2(x_2-x_1)=-F_1$$
$$m'_1(\ddot{x}'_1+\ddot{x}_g)+c'_1\dot{x}'_1+k'_1x'_1=F_1$$

$$\ldots$$

$$m_i(\ddot{x}_i+\ddot{x}_g)+c_i(\dot{x}_i-\dot{x}_{i-1})+k_i(x_i-x_{i-1})-c_{i+1}(\dot{x}_{i+1}-\dot{x}_i)-k_{i+1}(x_{i+1}-x_i)$$
$$-c'_{i+1}(\dot{x}'_{i+1}-\dot{x}_i)-k'_{i+1}(x'_{i+1}-x_i)=-F_i$$
$$m'_i(\ddot{x}'_i+\ddot{x}_g)+c'_i(\dot{x}'_i-\dot{x}_{i-1})+k'_i(x'_i-x_{i-1})=F_i$$

$$\ldots$$

$$m_N(\ddot{x}_N+\ddot{x}_g)+c_N(\dot{x}_N-\dot{x}_{N-1})+k_N(x_N-x_{N-1})=-F_N$$
$$m'_N(\ddot{x}'_N+\ddot{x}_g)+c'_N(\dot{x}'_N-\dot{x}_{N-1})+k'_N(x'_N-x_{N-1})=F_N$$

where $\ddot{x}_g$ is the ground acceleration; m_i, c_i and k_i are, respectively, the mass, the viscous damping and the stiffness coefficients of the i-th floor and m'_i, c'_i and k'_i are the corresponding values for the bracing connecting the dissipator and the main frame. F_i is the interaction force between the dissipator and the structure. The corresponding friction coefficients μ_i and the prestressing forces N_i limit the values of F_i:

$$F_i \leq \mu_i N_i \tag{3}$$

The set of $2N$ equations can be written in matrix form as:

$$\mathbf{M}^{ss}\ddot{\mathbf{x}}^s + \mathbf{C}^{ss}\dot{\mathbf{x}}^s + \mathbf{C}^{sd}\dot{\mathbf{x}}^d + \mathbf{K}^{ss}\mathbf{x}^s + \mathbf{K}^{sd}\mathbf{x}^d = -\mathbf{M}^{ss}\mathbf{r}\ddot{x}_g - \mathbf{F} \tag{4a}$$

$$\mathbf{M}^{dd}\ddot{\mathbf{x}}^d + (\mathbf{C}^{sd})^T\dot{\mathbf{x}}^s + \mathbf{C}^{dd}\dot{\mathbf{x}}^d + (\mathbf{K}^{sd})^T\mathbf{x}^s + \mathbf{K}^{dd}\mathbf{x}^d = -\mathbf{M}^{dd}\mathbf{r}\ddot{x}_g + \mathbf{F} \tag{4b}$$

Superscript s accounts for the structure and d for the dissipators: $\mathbf{x}^s = [x_1, x_2, \ldots, x_N]^T$ and $\mathbf{x}^d = [x'_1, x'_2, \ldots, x'_N]^T$. The second subset (4*b*) will be in its turn split in two subsets termed with subscripts *sl* (sliding) and *ns* (non-sliding); the degrees of freedom involved in each of them varies from instant to instant as the sliding conditions in the dissipators change. If the input consists of driving forces acting at every floor the right hand members of eqs (4*a*) and (4b) have to be replaced by **P** – **F** and **F**, respectively, where vector **P** contains the excitation forces.

Both matrix equations (4*a*) and (4*b*) can be put together as a $2N \times 2N$ system:

$$\begin{bmatrix}\mathbf{M}^{ss} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}^{dd}\end{bmatrix}\begin{bmatrix}\ddot{\mathbf{x}}^s\\ \ddot{\mathbf{x}}^d\end{bmatrix} + \begin{bmatrix}\mathbf{C}^{ss} & \mathbf{C}^{sd}\\ (\mathbf{C}^{sd})^T & \mathbf{C}\end{bmatrix}\begin{bmatrix}\dot{\mathbf{x}}^s\\ \dot{\mathbf{x}}^d\end{bmatrix} + \begin{bmatrix}\mathbf{K}^{ss} & \mathbf{K}^{sd}\\ (\mathbf{K}^{sd})^T & \mathbf{K}^{dd}\end{bmatrix}\begin{bmatrix}\mathbf{x}^s\\ \mathbf{x}^d\end{bmatrix} = -\begin{bmatrix}\mathbf{M}^{ss} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}^{dd}\end{bmatrix}\begin{bmatrix}\mathbf{r}\\ \mathbf{r}\end{bmatrix}\ddot{x}_g - \begin{bmatrix}\mathbf{F}\\ -\mathbf{F}\end{bmatrix} \tag{5}$$

4 Numerical solution of the equations of motion for multi-story frames

Blocks (4*a*) and (4*b*) are coupled through the interaction friction forces **F** and cross matrices $\mathbf{C}^{sd}$ (and $(\mathbf{C}^{sd})^T$) and $\mathbf{K}^{sd}$ (and $(\mathbf{K}^{sd})^T$).

At each generic instant $k+1$ the response is computed from the one at the previous instant k. This problem is numerically solved at each sampling instant k by a modification of the step-by-step linear acceleration method, where three nested iteration loops are performed. These iteration loops involve the coupling quantities ($\dot{\mathbf{x}}^d$, $\mathbf{x}^d$, $\dot{\mathbf{x}}^s$, $\mathbf{x}^s$ and **F**) and the estimated accelerations at the next step $k+1$ ($\ddot{\mathbf{x}}^{s*}_{k+1}$ and $\ddot{\mathbf{x}}^{d*}_{k+1}$)

It is initially assumed that the sliding conditions in the dissipators at instant k are kept for instant $k+1$. A set of values of $\ddot{\mathbf{x}}^{s*}_{k+1}$ (for $\ddot{\mathbf{x}}^{s}_{k+1}$), ${}_{sl}\ddot{\mathbf{x}}^{d*}_{k+1}$ (for ${}_{sl}\ddot{\mathbf{x}}^{d}_{k+1}$) and ${}_{ns}\mathbf{F}^{*}_{k+1}$ (for ${}_{ns}\mathbf{F}_{k+1}$) are assumed (usually $\ddot{\mathbf{x}}^{s*}_{k+1} = \ddot{\mathbf{x}}^{s}_{k}$, ${}_{sl}\ddot{\mathbf{x}}^{d*}_{k+1} = {}_{sl}\ddot{\mathbf{x}}^{d}_{k}$ and ${}_{ns}\mathbf{F}^{*}_{k+1} = {}_{ns}\mathbf{F}_{k}$). Remaining accelerations ${}_{ns}\ddot{\mathbf{x}}^{d}_{k+1}$ and forces ${}_{sl}\mathbf{F}_{k+1}$ will be equal to those of $\ddot{\mathbf{x}}^{s}_{k+1}$ and $\mathrm{sgn}(\dot{x}_i - \dot{x}'_i)\mu_i N_i$,

Transactions on the Built Environment vol 57, © 2001 WIT Press, www.witpress.com, ISSN 1743-3509

respectively. As stated in the previous paragraph, the proposed algorithm consists of three nested iteration loops. These iteration loops are performed with respect to $\ddot{\mathbf{x}}^s_{k+1}$, ${}_{ns}\mathbf{F}_{k+1}$ and ${}_{sl}\ddot{\mathbf{x}}^d_{k+1}$, respectively.

The interpolation criterion considered in the linear acceleration method [6,7] yields

$$\mathbf{x}^s_{k+1} = \mathbf{x}^s_k + \Delta t\dot{\mathbf{x}}^s_k + \frac{\Delta t^2}{6}\left(2\ddot{\mathbf{x}}^s_k + \ddot{\mathbf{x}}^{s*}_{k+1}\right) \tag{6a}$$

$$\dot{\mathbf{x}}^s_{k+1} = \dot{\mathbf{x}}^s_k + \frac{\Delta t}{2}\left(\ddot{\mathbf{x}}^s_k + \ddot{\mathbf{x}}^{s*}_{k+1}\right) \tag{6b}$$

$$\mathbf{x}^d_{k+1} = \mathbf{x}^d_k + \Delta t\dot{\mathbf{x}}^d_k + \frac{\Delta t^2}{6}\left(2\ddot{\mathbf{x}}^d_k + \ddot{\mathbf{x}}^{d*}_{k+1}\right) \tag{7a}$$

$$\dot{\mathbf{x}}^d_{k+1} = \dot{\mathbf{x}}^d_k + \frac{\Delta t}{2}\left(\ddot{\mathbf{x}}^d_k + \ddot{\mathbf{x}}^{d*}_{k+1}\right) \tag{7b}$$

The considered sliding conditions in the *i*-th dissipator (see eq (3)) are

$$\textit{If } F_i \geq \mu_i N_i \textit{ there is sliding and } F_i = \mathrm{sgn}(\dot{x}_i - \dot{x}'_i)\mu_i N_i \tag{8}$$

$$\textit{If } \left(\dot{x}^s_i - \dot{x}^d_i\right)_k \left(\dot{x}^s_i - \dot{x}^d_i\right)_{k+1} \leq 0 \textit{ the above sliding condition is checked} \tag{9}$$

The motion equations (4*a*) and (4*b*) (for the instants *k* and *k* + 1), the interpolation relationships (6) and (7) and the conditions (8) and (9) govern the motion of the building equipped with friction dissipators. The algorithm proposed to solve this problem is described next.

First iteration loop. Structure displacements $\mathbf{x}^s_{k+1}$ and velocities $\dot{\mathbf{x}}^s_{k+1}$ are computed from eqs (6*a*) and (6*b*). In the non-sliding dissipators the displacements ${}_{ns}\mathbf{x}^d_{k+1}$ and velocities ${}_{ns}\dot{\mathbf{x}}^d_{k+1}$ are equal to the corresponding values in $\mathbf{x}^s_{k+1}$ and $\dot{\mathbf{x}}^s_{k+1}$. Now $\ddot{\mathbf{x}}^s_{k+1}$ is computed from eq (4*a*) (for instant *k* + 1). If ${}_{sl}\ddot{\mathbf{x}}^{d*}_{k+1} \neq \ddot{\mathbf{x}}^{s*}_{k+1}$ (with a prescribed tolerance) this procedure is repeated taking $\ddot{\mathbf{x}}^{s*}_{k+1} = \ddot{\mathbf{x}}^s_{k+1}$ without making any changes in the sliding conditions (i e, dissipators keep their sliding condition). Once the convergence is reached, eq (4*a*) is fulfilled.

Second iteration loop. In the sliding dissipators the friction forces ${}_{sl}\mathbf{F}_{k+1}$ are known ($F_i = \mathrm{sgn}(\dot{x}_i - \dot{x}'_i)\mu_i N_i$) and the displacements ${}_{sl}\mathbf{x}^d_{k+1}$ and velocities ${}_{sl}\dot{\mathbf{x}}^d_{k+1}$ are computed from the corresponding equations (7*a*) and (7*b*) assuming ${}_{sl}\ddot{\mathbf{x}}^d_{k+1} = {}_{sl}\ddot{\mathbf{x}}^{d*}_{k+1}$. The friction forces ${}_{ns}\mathbf{F}_{k+1}$ in the presumed non-sliding dissipators are computed from the corresponding equations in eq (4*b*).

The components bigger than the corresponding sliding threshold $\mu_i N_i$ are set to that value and all of them are compared to the previously assumed values of ${}_{ns}\mathbf{F}^*_{k+1}$. If they are different (with a prescribed tolerance), the new values of ${}_{ns}\mathbf{F}^*_{k+1}$ are set equal to the calculated forces ${}_{ns}\mathbf{F}_{k+1}$. Then a new set of values of the structural acceleration $\ddot{\mathbf{x}}^s_{k+1}$ is computed from eq (4*a*) and it is replaced into (6*a*) and (6*b*) to get the updated values of $\mathbf{x}^s_{k+1}$ and $\dot{\mathbf{x}}^s_{k+1}$ (first loop). This procedure is repeated until ${}_{ns}\mathbf{F}_{k+1} = {}_{ns}\mathbf{F}^*_{k+1}$ (with a prescribed tolerance). Once the convergence is reached, eq (4*a*) and the non-sliding ones in (4*b*) are fulfilled.

Third iteration loop. Once the convergence in ${}_{ns}\mathbf{F}_{k+1}$ is achieved in the previous loop, then the accelerations ${}_{sl}\ddot{\mathbf{x}}^d_{k+1}$ are computed from the corresponding equations in eq (4*b*). If ${}_{sl}\ddot{\mathbf{x}}^d_{k+1} \neq {}_{sl}\ddot{\mathbf{x}}^{d*}_{k+1}$ (with a prescribed tolerance) the current values of ${}_{sl}\ddot{\mathbf{x}}^d_{k+1}$ are set equal to those of ${}_{sl}\ddot{\mathbf{x}}^{d*}_{k+1}$. After this, the values of the acceleration $\ddot{\mathbf{x}}^s_{k+1}$ are calculated from eq (4*a*). These values are used as the new approximations for the acceleration vector $\ddot{\mathbf{x}}^{s*}_{k+1}$ (first loop). The procedure stops when ${}_{sl}\ddot{\mathbf{x}}^d_{k+1} = {}_{sl}\ddot{\mathbf{x}}^{d*}_{k+1}$ (with a prescribed tolerance). Once the convergence is reached, eqs (4*a*) and (4*b*) (both for sliding and non-sliding components) are fulfilled.

5 Results

In this section some figures are shown to highlight the results obtained with the proposed algorithm and to assert the usefulness of dissipators to reduce significantly the dynamic response of buildings.

Figure 3 shows the time-history response of a single-story frame (i e, N = 1) equipped with a friction dissipator (see Figure 1) and subjected to a harmonic driving force $P(t) = P0 \sin \omega t$. The mass, damping and stiffness are m = 5.8534 kg·sec^2/cm, c = 13.3800 kg·sec^2/cm, k = 3058.50 kg/cm; m' = 0.02 kg·sec^2/cm, c' = 0.0, k' = 2648.1201 kg/cm; sliding threshold is μN = 4009.91 kg. The amplitude and frequency of the excitation are $P0$ = 13000 kg and $\omega = 15.9781$ rad/sec. In the calculation process, an increment of time Δt = 0.001150488 sec was used.

Continuos plot in Figure 3 shows the response given by the algorithm proposed, and the dotted plot shows the response given by the ADINA program. The comparison between them shows a good agreement.

Figures 4 and 5 show two time-history responses of a 3-story building frame (N = 3) undergoing seismic excitation. The frame is modelled as a shear building. The mass, damping, and stiffness matrices are, respectively

$$\mathbf{M} = \begin{bmatrix} 0.2590 & 0 & 0 \\ 0 & 0.2590 & 0 \\ 0 & 0 & 0.1295 \end{bmatrix} \text{kips·sec}^2\text{/in} \qquad \mathbf{C} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{kips·sec/in}$$

$$\mathbf{K} = \begin{bmatrix} 250 & -100 & 0 \\ -100 & 150 & -50 \\ 0 & -50 & 50 \end{bmatrix} \text{kips/in}$$

The corresponding matrices for the bracing-dissipator system are

$$\mathbf{M}' = \begin{bmatrix} 0.001 & 0 & 0 \\ 0 & 0.001 & 0 \\ 0 & 0 & 0.0005 \end{bmatrix} \text{kips·sec}^2\text{/in} \qquad \mathbf{C}' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{kips·sec/in}$$

$$\mathbf{K}' = \begin{bmatrix} 212.5 & 0 & 0 \\ 0 & 127.5 & 0 \\ 0 & 0 & 42.50 \end{bmatrix} \text{kips/in}$$

The sliding thresholds are $\mu_1 N_1 = 40$ kips, $\mu_2 N_2 = 50$ kips and $\mu_3 N_3 = 20$ kips. The seismic input is the 'Imperial Valley' earthquake, 'El Centro' register (May 18, 1940, 270°). The maximum acceleration is 134.53 in/sec^2. A factor of 5 was used to amplify the ordinates.

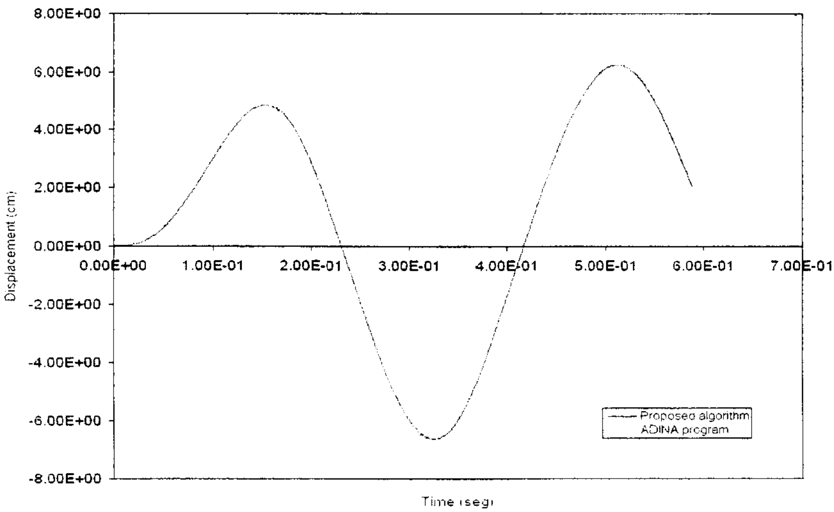

Figure 3: Comparison of time-history responses obtained with the proposed algorithm and the ADINA program for a single-story building subjected to a periodic load

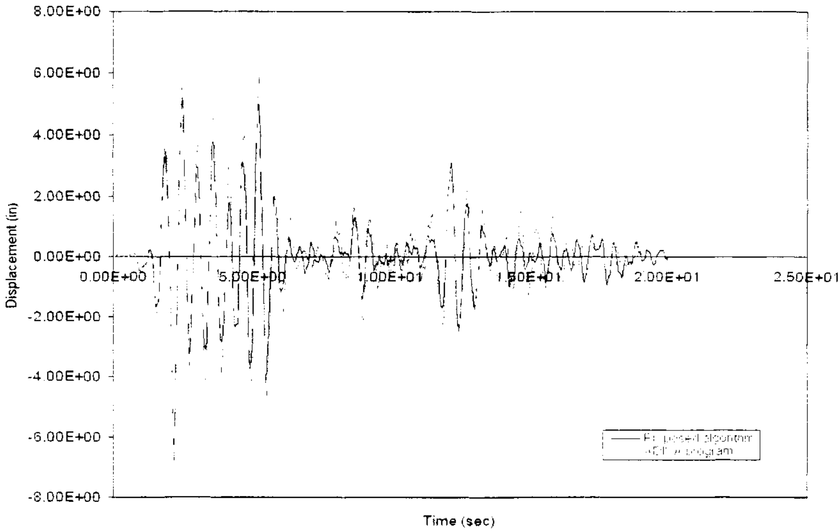

Figure 4: Comparison of the relative displacements calculated with ADINA and the proposed algorithm for the first floor for a ground acceleration

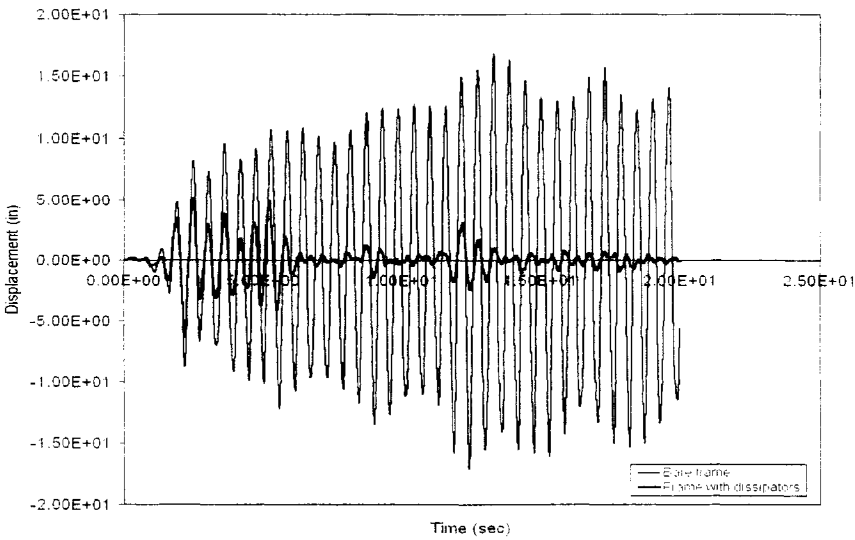

Figure 5: Comparison of the relative displacements with and without dissipators for the first floor for a ground acceleration

Plots from Figure 5 highlight the reduction in the structural response due to the dissipation effect of the friction devices.

6 Conclusions and future research

In this paper a numerical model of the dynamic behavior of frames modelled as MDOF systems and incorporating friction energy dissipators is presented. A step-by-step numerical algorithm developed by the authors solves the nonlinear equations of motion. This algorithm is based on the linear acceleration method (Newmark's method). Results are close to those obtained with the commercial package ADINA. Initial studies tend to show that dissipators significantly reduce the response of the structure.

Currently, a parametric study about structures that incorporate this type of passive dampers is being carried out. This research includes experimental testing.

Acknowledgments

This work has received financial support from the Spanish Government (DGICYT), Research Projects No. PB98-0455, AMB980558 and PB96-0139. The stay of Mr. De la Cruz at Barcelona is supported by the Council of Science and Technology (CONACY) of Mexico, grant No. 116708/117876 and by the Spanish Agency of International Cooperation (AECI) of Spain.

References

[1] T. T. Soong and G. F. Dargush, *Passive Energy Dissipation Systems in Structural Engineering,* John Wiley & Sons: England, 1997.
[2] R. Hanson, I. Aiken, D. Nims, P. Richter, and R. Bachmann, State of the art of the practice in seismic energy dissipation. *Proc. ATC 17-1 on Seismic Isolation, Energy Dissipation and Active Control,* 2, pp. 449–471, 1993.
[3] K. J. Bathe, *Finite Element Procedures in Engineering Analysis,* Prentice-Hall Inc.: Englewood Cliffs, N. J., 1982.
[4] J. Oliver, S. Oller and J. C. Cante, Numerical simulation of uniaxial compaction processes in powder materials. *International Congress on Numerical Methods in Engineering:* Chile, 1992.
[5] A. K. Chopra, *Dynamics of Structures, Theory and applications to earthquake engineering*, Prentice-Hall, Inc.: Englewood Cliffs, N. J., 2nd ed., 2001.
[6] G. Berg, *Elements of Structural Dynamics,* Prentice-Hall, Inc.: Englewood Cliffs, N. J., 1989.
[7] R. Clough and J. Penzien, *Dynamics of Structures,* McGraw-Hill: New York, 1992.