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Proper defect density parameters for anisotropic solids with cracks and elliptical holes

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Abstract

Effective elastic moduli of a *2-D* anisotropic solid with elliptical holes and cracks having an arbitrary (non-random) orientational distribution are given in closed form. Proper tensorial parameters of defect density (dependent on ellipses' eccentricity and their orientations *relative* to the matrix anisotropy axes) are identified. This allows one to establish the overall elastic anisotropy. The results for *mixtures* of holes and cracks are presented.

1 Introduction

We call the density parameter *proper* if it correctly reflects the individual hole contributions to the overall elastic compliance. Only in terms of such a parameter can the effective compliances be uniquely expressed. Identification of the proper density parameter is a non-trivial problem, since the individual hole contributions to the overall compliances depend not only on the hole shapes but on their orientations *relative* to the matrix anisotropy axes (for example, elongated holes normal to the "stiffer" direction of the matrix produce a higher impact on the overall compliance than the ones normal to the "softer" direction). We show, following Kachanov *et al.* [1], that the proper density parameter is implied by the structure of the elastic potential. The potential-based procedure to derive the effective compliances of an anisotropic material with holes and cracks is outlined below.

Strain per reference area A containing a hole is a sum: $\boldsymbol{\varepsilon} = \mathbf{S}^0 : \boldsymbol{\sigma} + \Delta\boldsymbol{\varepsilon}$ (or $\varepsilon_{ij} = S^0_{ijkl}\sigma_{kl} + \Delta\varepsilon_{ij}$ in indicial notation), where $\mathbf{S}^0$ is the compliance tensor of the matrix. Hole's contribution $\Delta\boldsymbol{\varepsilon}$ is given by the integral

$$\Delta\boldsymbol{\varepsilon} = -(1/2A)\int_{\Gamma}(\mathbf{un} + \mathbf{nu})\,d\Gamma \tag{1}$$

where $\mathbf{n}$ is the unit normal to hole boundary Γ (inwards the hole) and $\mathbf{u}$ denotes displacements of points of Γ and $\mathbf{un}$, $\mathbf{nu}$ are dyadic products of two vectors.

Due to linear elasticity, $\Delta\boldsymbol{\varepsilon}$ is a linear function of $\boldsymbol{\sigma}$ and hence can be written as

$$\Delta\boldsymbol{\varepsilon} = \mathbf{H}:\boldsymbol{\sigma} \tag{2}$$

where fourth rank tensor $\mathbf{H}$ is the *cavity compliance tensor* (possessing the usual symmetries $H_{ijkl} = H_{jilk} = H_{klij}$). $\mathbf{H}$-tensors were found for a number of *2-D* and *3-D* cavity shapes by Tsukrov and Kachanov [2] and Kachanov *et al.* [1], in the case of the *isotropic* matrix. Mauge and Kachanov [3] derived results for an *anisotropic* matrix with *cracks* of arbitrary orientations. Here, we present the results for holes, cracks and *mixtures* of holes and cracks in the orthotropic matrix.

We formulate the problem in terms of the elastic *potential* (rather than compliances): its structure implies the proper parameters of defect density and establishes the overall anisotropy. The potential in stresses $f(\boldsymbol{\sigma}) = (1/2)\boldsymbol{\sigma}:\boldsymbol{\varepsilon}(\boldsymbol{\sigma})$ of a solid with a hole is

$$f(\boldsymbol{\sigma}) = (1/2)\boldsymbol{\sigma}:\mathbf{S}^0:\boldsymbol{\sigma} + (1/2)\boldsymbol{\sigma}:\mathbf{H}:\boldsymbol{\sigma} \equiv f_0 + \Delta f \tag{3}$$

where f_0 is the elastic potential of the matrix material, Δf is the change in the potential due to the presence of a hole. For the *orthotropic* solid,

$$f_0(\sigma_{ij}) = (1/2E_1^0)\sigma_{11}^2 + (1/2E_2^0)\sigma_{22}^2 - (\nu_{12}^0/E_1^0)\sigma_{11}\sigma_{22} + (1/2G_{12}^0)\sigma_{12}^2$$

where E_i^0, G_{ij}^0, and ν_{ij}^0 are Young's moduli, shear moduli and Poisson's ratios of the matrix in the case of plane stress. In plane strain, $E_1 \to E_1/(1-\nu_{13}\nu_{31})$, $\nu_{12} \to (\nu_{12} + \nu_{13}\nu_{32})/(1-\nu_{13}\nu_{31})$, $E_2 \to E_2/(1-\nu_{23}\nu_{32})$, $\nu_{21} \to (\nu_{21} + \nu_{23}\nu_{31})/(1-\nu_{23}\nu_{32})$.

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In the case of a *crack*, expression (1) reduces to $\Delta\varepsilon = \left(l^2/A\right)(\mathbf{bn} + \mathbf{nb})$, where vector $\mathbf{b} = \left\langle \mathbf{u}^+ - \mathbf{u}^- \right\rangle / l$ is the average over the crack displacement discontinuity normalized to the crack length l. Due to linear elasticity, $\mathbf{b}$ is a linear function of traction $\mathbf{n} \cdot \boldsymbol{\sigma}$ induced by remotely applied $\boldsymbol{\sigma}$ at the crack site in a continuous material:

$$\mathbf{b} = \mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{B} \qquad (\text{or } b_i = \sigma_{kj} n_k B_{ji}) \tag{4}$$

where symmetric second rank tensor $\mathbf{B}$ is the *crack compliance tensor*. The change in the potential due to *2-D* rectilinear crack is

$$\Delta f = (1/2)\boldsymbol{\sigma}{:}\Delta\varepsilon = (1/A)\boldsymbol{\sigma}{:}\, l^2 \mathbf{nBn} : \boldsymbol{\sigma} \tag{5}$$

hence identifying $\mathbf{H}$ -tensor of a crack as (appropriately symmetrized)

$$\mathbf{H} = \left(2l^2/A\right)\mathbf{nBn} \tag{6}$$

or $H_{ijkl} = \left(l^2/2\right)\left(n_i B_{jk} n_l + n_j B_{ik} n_l + n_i B_{jl} n_k + n_j B_{il} n_k\right)$.

We now consider a matrix with *many* holes in the *non-interaction approximation*: each hole is placed into $\boldsymbol{\sigma}$ -field and does not experience any influence of neighbors. This approximation is of the fundamental importance: besides being rigorous at small defect densities, it constitutes the basic building block for various approximate. In the non-interaction approximation, the potential change due to holes is

$$\Delta f = (1/2)\boldsymbol{\sigma}{:}\sum \mathbf{H}^{(k)}{:}\boldsymbol{\sigma} \tag{7}$$

Tensor $\mathbf{H} = \sum \mathbf{H}^{(k)}$ (summation may be replaced by integration over orientations, if computationally convenient) takes the individual cavity contributions with proper "relative weights" and, thus, constitutes *the proper parameter of defect density*. Below, we specialize it for elliptical holes in the orthotropic matrix. The effective elastic compliances S_{ijkl} are obtained from $\varepsilon_{ij} = \partial(f_0 + \Delta f)/\partial\sigma_{ij} \equiv S_{ijkl}\sigma_{kl}$.

In the case of *cracks*, equation (7) takes the form

$$\Delta f = \boldsymbol{\sigma} : (1/A)\sum \left(l^2 \mathbf{nBn}\right)^{(k)} : \boldsymbol{\sigma} \tag{8}$$

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This introduces fourth rank tensor $(1/A)\sum\left(l^2\mathbf{nBn}\right)^{(k)}$, appropriately symmetrized, as the proper general parameter of *crack density*. For the *isotropic* matrix, $\mathbf{B}=(\pi/E)\mathbf{I}$, where $\mathbf{I}$ is the unit tensor; then $\Delta f=(\pi/E)\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}:\boldsymbol{\alpha}$ thus identifying symmetric second rank crack density tensor (Kachanov [4]) as the proper parameter of crack density:

$$\boldsymbol{\alpha}=(1/A)\sum\left(l^2\mathbf{nn}\right)^{(k)} \tag{9}$$

2 An elliptical hole in the orthotropic matrix

For an elliptical hole arbitrary oriented in the orthotropic matrix (Fig. 1), components of the hole compliance tensor can be derived using results of Lekhnitski [5] as follows

$$\left.\begin{aligned}
H_{1111}&=\frac{\pi L}{A\sqrt{E_1^0}}\left[\left(b^2-a^2\right)\cos^2\varphi+a^2+\frac{ab}{L\sqrt{E_1^0}}\right]\\
H_{1112}&=\frac{\pi\left(b^2-a^2\right)L}{2A\sqrt{E_1^0}}\sin\varphi\cos\varphi,\qquad H_{1122}=-\frac{\pi ab}{A\sqrt{E_1^0E_2^0}}\\
H_{1212}&=\frac{\pi L}{4A\sqrt{E_1^0E_2^0}}\left[\left(a^2-b^2\right)\left(\sqrt{E_2^0}-\sqrt{E_1^0}\right)\cos^2\varphi\right.\\
&\qquad\left.+a^2\sqrt{E_1^0}+abL\sqrt{E_1^0E_2^0}+b^2\sqrt{E_2^0}\right]\\
H_{1222}&=\frac{\pi\left(b^2-a^2\right)L}{2A\sqrt{E_2^0}}\sin\varphi\cos\varphi,\\
H_{2222}&=\frac{\pi L}{A\sqrt{E_2^0}}\left[\left(a^2-b^2\right)\cos^2\varphi+b^2+\frac{ab}{L\sqrt{E_2^0}}\right]
\end{aligned}\right\} \tag{10}$$

where $L=\sqrt{1/G_{12}^0-2\nu_{12}^0/E_1^0+2/\sqrt{E_1^0E_2^0}}$.

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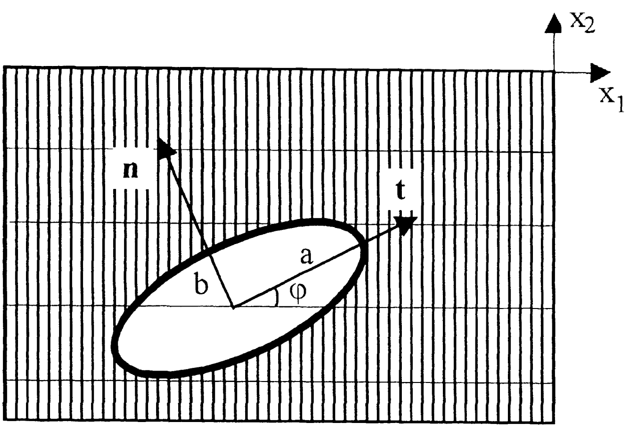

Figure 1: Elliptical hole in the orthotropic matrix (x_1, x_2 are the orthotropy axes with unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$).

In the case of a *crack* ($b = 0$, $a = l$), the dependence of $\mathbf{H} = \mathbf{nBn}$ on the crack orientation φ is remarkably simple: it is expressed only through vector $\mathbf{n}$ whereas *crack compliance tensor* $\mathbf{B}$ *is constant* (independent of φ):

$$\mathbf{B} = \frac{\pi C}{2}(1 + D)\mathbf{e}_1\mathbf{e}_1 + \frac{\pi C}{2}(1 - D)\mathbf{e}_2\mathbf{e}_2 \qquad (11)$$

where

$$C = \frac{1}{2}\frac{\sqrt{E_1^0} + \sqrt{E_2^0}}{\sqrt{E_1^0 E_2^0}}\sqrt{\frac{1}{G_{12}^0} - \frac{2\nu_{12}^0}{E_1^0} + \frac{2}{\sqrt{E_1^0 E_2^0}}}, \quad D = \frac{\sqrt{E_1^0} - \sqrt{E_2^0}}{\sqrt{E_1^0} + \sqrt{E_2^0}}.$$

In the case of a *circular hole* ($a = b$), hole's *influence on the elastic compliance is anisotropic*: the reduction of stiffness due to the hole is greater in the stiffer direction of the matrix (in spite of the "geometrically isotropic" shape of the hole). This implies that circular holes reduce the extent of the matrix anisotropy.

3 One family of parallel ellipses

Let us consider one family of parallel ellipses of an arbitrary orientation φ. This case is important since, for any orientational distribution of non-interacting ellipses, the effective elastic compliances are obtained by integration over orientations (with appropriate distribution density) of the results of this section.

Tensor $\mathbf{H}^* = \sum \mathbf{H}^{(k)}$ is expressed in terms of scalar porosity $p = (1/A)\pi \sum (ab)^{(k)}$ and symmetric second rank *hole density tensor* $\boldsymbol{\beta} = (1/A)\pi \sum \left(a^2 \mathbf{nn} + b^2 \mathbf{tt}\right)^{(k)}$ as follows

$$\left.\begin{aligned}
&H^*_{1111} = \frac{1}{E_1^0}\left[p + L\sqrt{E_1^0}\left(\beta_{nn}\sin^2\varphi + \beta_{tt}\cos^2\varphi\right)\right]\\
&H^*_{1112} = \frac{L}{2\sqrt{E_1^0}}\left(\beta_{tt} - \beta_{nn}\right)\sin\varphi\cos\varphi\\
&H^*_{1212} = \frac{L}{4\sqrt{E_1^0 E_2^0}}\left[pL\sqrt{E_1^0 E_2^0} + \sqrt{E_1^0}\left(\beta_{nn}\sin^2\varphi + \beta_{tt}\cos^2\varphi\right)\right.\\
&\qquad\left. + \sqrt{E_2^0}\left(\beta_{nn}\cos^2\varphi + \beta_{tt}\sin^2\varphi\right)\right]\\
&H^*_{1122} = -\frac{p}{\sqrt{E_1^0 E_2^0}}, \qquad H^*_{1222} = \frac{L}{2\sqrt{E_2^0}}\left(\beta_{tt} - \beta_{nn}\right)\sin\varphi\cos\varphi\\
&H^*_{2222} = \frac{1}{E_2^0}\left[p + L\sqrt{E_2^0}\left(\beta_{nn}\cos^2\varphi + \beta_{tt}\sin^2\varphi\right)\right]
\end{aligned}\right\} \tag{12}$$

where $\beta_{nn} = (1/A)\pi \sum a^{(k)2}$ and $\beta_{tt} = (1/A)\pi \sum b^{(k)2}$ are components of $\boldsymbol{\beta}$.

The effective elastic constants are, generally, non-orthotropic (except for the cases $\varphi = 0$ or $\varphi = \pi/2$). Some of the elastic moduli, given in terms of the "engineering constants" are

$$\left.\begin{aligned}
&\left(E_1/E_1^0\right)^{-1} = 1 + p + L\sqrt{E_1^0}\left(\beta_{nn}\sin^2\varphi + \beta_{tt}\cos^2\varphi\right)\\
&\left(E_2/E_2^0\right)^{-1} = 1 + p + L\sqrt{E_2^0}\left(\beta_{nn}\cos^2\varphi + \beta_{tt}\sin^2\varphi\right)\\
&\nu_{12}/E_1 = \nu_{12}^0/E_1^0 + p\Big/\sqrt{E_1^0 E_2^0}
\end{aligned}\right\} \tag{13}$$

In the case of matrix *isotropy*, results of Kachanov *et al.* [1] are recovered.

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Results for *circular* holes are obtained as a special case, at $\boldsymbol{\beta} = p\mathbf{I}$. The dependence on φ vanishes and we obtain, for example, $\left(E_1/E_1^0\right)^{-1} = 1 + p\left(1 + L\sqrt{E_1^0}\right)$.

4 Randomly oriented ellipses

In this case, components of hole density tensor can be expressed in terms of porosity p and the *eccentricity parameter* $q = (\pi/A)\sum (a-b)^{2(k)}$ as follows

$$\left.\begin{aligned}
H_{1111}^* &= \frac{1 + L\sqrt{E_1^0}}{E_1^0} p + \frac{L}{2\sqrt{E_1^0}} q \\
H_{1122}^* &= -\frac{1}{\sqrt{E_1^0 E_2^0}} p \\
H_{1212}^* &= \frac{L}{8\sqrt{E_1^0 E_2^0}}\left[2\left(\sqrt{E_2^0} + L\sqrt{E_1^0 E_2^0} + \sqrt{E_1^0}\right)p + \left(\sqrt{E_2^0} + \sqrt{E_1^0}\right)q\right] \\
H_{2222}^* &= \frac{1 + L\sqrt{E_2^0}}{E_2^0} p + \frac{L}{2\sqrt{E_2^0}} q
\end{aligned}\right\}$$

In spite of random orientations, tensor $\mathbf{H}$ is not isotropic: randomly oriented ellipses produce a higher impact on the compliance in the "stiffer" direction of the matrix and, thus, reduce the overall anisotropy. The effective moduli are orthotropic and given by

$$\frac{E_1}{E_1^0} = \left[1 + \left(1 + L\sqrt{E_1^0}\right)p + (L/2)\sqrt{E_1^0}\, q\right]^{-1}, \quad \frac{\nu_{12}}{E_1} = \frac{\nu_{12}^0}{E_1^0} + \frac{p}{\sqrt{E_1^0 E_2^0}}$$

$$\frac{E_2}{E_2^0} = \left[1 + \left(1 + L\sqrt{E_2^0}\right)p + (L/2)\sqrt{E_2^0}\, q\right]^{-1} \qquad (14)$$

$$\frac{G_{12}}{G_{12}^0} = \left\{1 + \frac{LG_{12}^0}{4\sqrt{E_1^0 E_2^0}}\left[\left(\sqrt{E_2^0} + L\sqrt{E_1^0 E_2^0} + \sqrt{E_2^0}\right)p + \left(\sqrt{E_2^0} + \sqrt{E_2^0}\right)q\right]\right\}^{-1}$$

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We emphasize that, even in the case of random orientations, the moduli cannot be expressed in terms of porosity p alone – a second hole density parameter q is also needed.

Note gradual disappearance of anisotropy as the hole density increases. The ratio of Young's moduli E_1/E_2 characterizing the extent of anisotropy is:

$$\frac{E_1}{E_2} = \frac{E_1^0}{E_2^0}\,\frac{1+\left(1+L\sqrt{E_2^0}\right)p+(L/2)\sqrt{E_2^0}\,q}{1+\left(1+L\sqrt{E_1^0}\right)p+(L/2)\sqrt{E_1^0}\,q} \tag{15}$$

5 Mixtures of holes and cracks

The approach presented in this paper covers, in a unified way, any mixtures of holes of diverse eccentricities (including cracks). For example, in the case of a mixture of circular holes and cracks, the hole density tensor $\boldsymbol{\beta}$ can be represented as a sum

$$\boldsymbol{\beta} = \boldsymbol{\beta}_{\text{holes}} + \boldsymbol{\beta}_{\text{cracks}} = p\mathbf{I} + \boldsymbol{\alpha} \tag{16}$$

Below, the results for the mixture of circular holes and parallel cracks (Figs 2, 3) are presented. The effective elastic moduli are expressed in terms of porosity p and *scalar crack density* $\rho = (\pi/A)\sum l^{(k)2}$ (linear invariant of the crack density tensor $\boldsymbol{\alpha}$, see eqn (9)) as follows

$$\left.\begin{aligned}
\left(E_1/E_1^0\right)^{-1} &= 1+\rho\pi L\sqrt{E_1^0}\sin^2\varphi + p\left(1+L\sqrt{E_1^0}\right)\\
\left(E_2/E_2^0\right)^{-1} &= 1+\rho\pi L\sqrt{E_2^0}\cos^2\varphi + p\left(1+L\sqrt{E_2^0}\right)\\
\left(G_{12}/G_{12}^0\right)^{-1} &= 1+\frac{\rho\pi G_{12}^0 L}{2}\left(\cos^2\varphi\Big/\sqrt{E_1^0}+\sin^2\varphi\Big/\sqrt{E_2^0}\right)\\
&\quad+\frac{pG_{12}^0 L}{2}\left(1\Big/\sqrt{E_1^0}+L+1\Big/\sqrt{E_2^0}\right)\\
\nu_{12}/E_1 &= \nu_{12}^0/E_1^0 + p\Big/\sqrt{E_1^0E_2^0}
\end{aligned}\right\} \tag{17}$$

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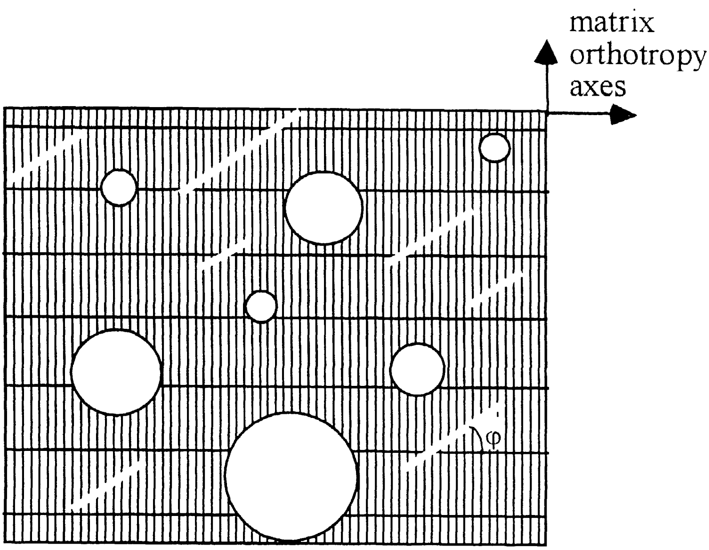

Figure 2: Mixture of circular holes and parallel cracks

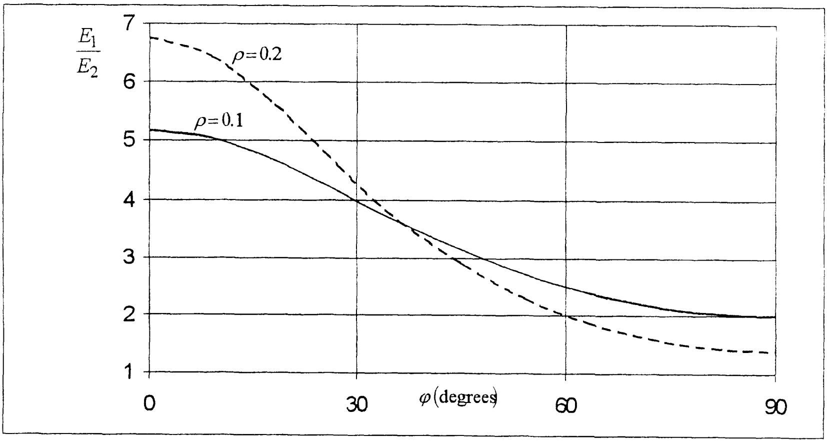

Figure 3: Ratio E_1/E_2 for the orthotropic matrix ($E_1^0/E_2^0 = 4.1$, $E_1^0/G_{12}^0 = 10$, $\nu_{12}^0 = 0.277$, $\nu_{21}^0 = 0.068$) with circular holes having porosity $p = 0.1$ and cracks inclined at angle φ to x_1-axis

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6 Conclusions

Effective elastic properties of an anisotropic matrix with elliptical holes of an arbitrary orientational distribution are derived in closed form. One of the key problems is the identification of the proper parameters of defect density that correctly reflect the individual defect contributions to the overall compliances. Expressions of the effective moduli in terms of such parameters apply to all orientational distributions of holes. Another advantage of the proposed approach is that any mixtures of holes of diverse eccentricities (including cracks) are covered in a unified way.

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