# New formulas for the motion resistance of debris flows

D. Berzi[1], J. T. Jenkins[2] & E. Larcan[1]
[1]*Department of Environmental, Hydraulic, Infrastructures and Surveying Engineering, Politecnico di Milano, Italy*
[2]*School of Civil and Environmental Engineering, Cornell University, USA*

## Abstract

We simplify a two-phase theory proposed by Berzi and Jenkins for the uniform motion of a granular-fluid mixture to obtain explicit, analytical relations between the tangent of the angle of inclination of the free surface, the average particle (fluid) velocity and the particle (fluid) depth. Those expressions, valid, in principle, only in uniform flow conditions, can then be employed to express the motion resistance for the particles and the fluid in mathematical models of non-uniform flow, as customary in Hydraulics. The advantages of those formulas with regard to previous, widely employed expressions are also discussed.
*Keywords: rheology, uniform flow, friction slope.*

## 1 Introduction

Recently, Berzi and Jenkins [1–3] proposed a simple theory based on a linear rheology for the particle interactions, turbulent shearing of the fluid, buoyancy, and drag. They provided a complete analytical description of the steady, uniform flow of a granular-fluid mixture (debris flow) over an inclined bed contained between frictional sidewalls. In order to obtain such analytical solution, they assumed a constant concentration in the particle-fluid mixture and the similarity of the particle and fluid velocity profiles. The predictions of this description compared favourably with the measurements in experiments on steady, uniform granular-fluid flows performed by Armanini *et al.* [4] and Larcher *et al.* [5] on mono-dispersed plastic cylinders and water. As seen in the experiments, the particle and fluid velocity distributions, the flow depths, and the free surface

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inclination were completely determined by the particle and fluid volume fluxes. Here, we simplify the theory of Berzi and Jenkins [1–3] by neglecting the turbulent shear stress in the mixture and the presence of the sidewalls. We can therefore obtain explicit relations between the average particle velocity, the depth and the tangent of the angle of inclination of free surface and between the average fluid velocity, the depth and the tangent of the angle of inclination of free surface. Those relations can then be used as analytical expressions of the motion resistance encountered by the particles and the fluid, respectively, in a debris flow, by interpreting the angle of inclination of the free surface as the so called friction slope.

The paper is organized as follows: first, we briefly recall the theory of Berzi and Jenkins [1–3]; then, we derive simplified expressions for the friction slopes and, finally, discuss them in comparison with other well-known formulas.

## 2 Theory

We let $\rho$ denote the fluid mass density, $c$ the particle concentration, $g$ the gravitational acceleration, $\sigma$ the particle specific mass, $d$ the particle diameter, $\eta$ the fluid viscosity, $U$ the fluid velocity, and $u$ the particle velocity. The Reynolds number $R = \rho d(gd)^{1/2}/\eta$ characterizes the fall velocity of the particles. In what follows, we phrase the momentum balances and constitutive relations in terms of dimensionless variables, with lengths made dimensionless by $d$, velocities by $(gd)^{1/2}$, and stresses by $\rho\sigma gd$.

We take $z = 0$ to be the top of the grains, $z = h$ to be the position of the rigid bed, and $H$ to be the height of the water above a bed of inclination $\theta$. The degree of saturation, $\xi = H/h$, is greater than unity in the over-saturated case and less than unity in the under-saturated. Sketches of over- and under-saturated flows are depicted in figure 1, together with a generic velocity profile for the particles.

We assume that it is possible to apply the rheology proposed by the French group GDR MiDi [6]. This rheology provides the particle stress ratio $\mu \equiv s/p$ and the concentration $c$ as unique functions of the inertial parameter

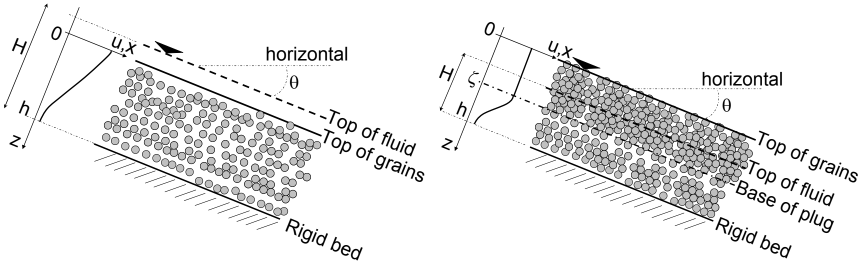


Figure 1: Sketch of steady, (a) over- and (b) under-saturated, uniform flows over rigid beds.

$I \equiv |\dot{\gamma}| / (p/c)^{1/2}$, where $s$ is the particle shear stress, $p$ the particle effective pressure and $\dot{\gamma}$ is the strain rate. In this case, $|\dot{\gamma}| = -u'$; where here, and in what follows, a prime indicates a derivative with respect to $z$.

We consider highly concentrated flows, in which the functions are approximately linear [7],

$$\mu = \breve{\mu} + \chi I \tag{1}$$

and $c = \hat{c} - bI$, where $\breve{\mu}$ and $\hat{c}$ are the minimum stress ratio and the maximum concentration, respectively, and $\chi$ and $b$ are material coefficients. The quantities $\breve{\mu}$ and $\hat{c}$ characterize both the bed and the plug, at which $I = 0$; $\breve{\mu}$ is the tangent of the angle of repose and $\hat{c}$ is the concentration at dense, random packing.

The balances of fluid momentum transverse and parallel to the flow, in the region in which both phases are present, are

$$P' = \cos\theta / \sigma, \tag{2}$$

and

$$S' = (1-c)\sin\theta/\sigma - cC(U-u)/\sigma, \tag{3}$$

respectively, where $P$ is the fluid pressure, $S$ the fluid shear stress, and $C$ is the dimensionless drag,

$$C \equiv (3|U-u|/10 + 18.3/R)/(1-c)^{3.1}, \tag{4}$$

derived by Dallavalle [8], with the concentration dependence suggested by Richardson and Zaki [9]. When an upper clear fluid layer is present, the distribution of the fluid shear stress can be obtained from eqn. (3) with $c = 0$.
The balances of particle momentum transverse and parallel to the flow are

$$p' = (1 - 1/\sigma) c \cos\theta, \tag{5}$$

and

$$s' = c\sin\theta + cC(U-u)/\sigma, \tag{6}$$

respectively. The balances for the particles when an upper dry layer is present can be obtained from eqns. (5) and (6) by letting $\sigma$ become infinite.

Here, in the mixture, we ignore the turbulent shear stress in the fluid relative to gravity and drag and neglect the friction of the sidewalls. In the clear fluid layer, we assume that the turbulent mixing length is proportional to the thickness of the layer:

$$S = -k^2 (H-h)^2 \left|U'\right| U', \tag{7}$$

where $k = 0.20$, half the value of Karman's constant. We also assume that the concentration is approximately constant and at its maximum value, $c = \hat{c}$.
With these assumptions, and considering the surface at $z = 0$ as free of particle stress, it is possible to obtain the particle stress ratio, $\mu$, as a function of $z$ from the momentum balances (2), (3), (5) and (6):

$$\mu = \frac{\sigma z + (1-\hat{c})\left[z - h(1-\alpha)\right]/\hat{c}}{\sigma z - z + h(1-\alpha)} \tan\theta + \frac{\sigma S^*}{\left[\sigma z - z + h(1-\alpha)\right]\hat{c}\cos\theta} \tag{8}$$

(for details of this derivation, see [2]), where $\alpha = H/h$ in an under-saturated flow and unity otherwise; and $S^* = h(\beta - 1)\sin\theta/\sigma$ is the fluid shear stress at the top of the particles, where $\beta = H/h$ in an over-saturated flow and unity otherwise.

## 2.1 Particles

In the upper dry layer, $\mu$ is constant and equal to $\tan\theta$ (from eqn. 9, with $\sigma$ equal to infinity). Given the linear rheology (1), in the under-saturated flows, the upper dry layer is either totally sheared, when $\tan\theta > \breve{\mu}$, or there is a plug in the region $\zeta \le z \le 0$. The location $\zeta$ of the base of the plug can be found from eqn. (8), with $\mu = \breve{\mu}$ and $S^* = 0$:

$$\frac{\zeta}{h} = \frac{\left[(1-\hat{c})\tan\theta + \hat{c}\breve{\mu}\right](1-\alpha)}{\hat{c}(\sigma - 1)(\tan\theta - \breve{\mu}) + \tan\theta}. \tag{9}$$

For reasonable values of $\tan\theta$, eqn. (9) can be approximated by $\zeta/h = 1-\alpha$. In this case, the average particle velocity along $h$ is simply equal to

$$u_A = u_m \alpha + u_{dry}(1-\alpha), \tag{10}$$

where $u_m$ is the mean particle velocity in the mixture layer and $u_{dry}$ the mean particle velocity in the dry layer. The quantity $u_m$ can be obtained once known the velocity distribution in the mixture layer. The latter can be obtained using eqn. (1) in eqn. (9), with $I = -\sigma u' / \left\{\left[(\sigma-1)z + h(1-\alpha)\right]\cos\theta\right\}^{1/2}$, and integrating:

$$\frac{u}{(1-1/\sigma)^{1/2}} = -\frac{2}{3}\frac{(z+L)^{1/2}}{\chi}A\left[(z+L)-3(-N/A+L)\right] + \frac{2}{3}\frac{(h+L)^{1/2}}{\chi}A\left[(h+L)-3(-N/A+L)\right], \tag{11}$$

where $A=\left[\sigma+(1-\hat{c})/\hat{c}\right]\tan\theta/(\sigma-1)-\breve{\mu}$, $L=h(1-\alpha)/(\sigma-1)$, and $N=\left[-(1-\hat{c})\tan\theta/\hat{c}-\breve{\mu}\right]L+h(\beta-1)\tan\theta/(\hat{c}\sigma-\hat{c})$. In obtaining eqn. (11), we have assumed a mild slope, so that $\cos\theta\approx 1$, and a zero slip velocity at the bed. It is then possible to obtain $\alpha u_m$ by integrating eqn. (11) between (1-$\alpha$)$h$ and $h$:

$$\alpha u_m = \frac{2}{3}\frac{(\sigma-1)^{1/2}h^{3/2}}{\sigma^{1/2}\chi}\left\{\left[\left(\frac{\sigma-\alpha}{\sigma-1}\right)^{3/2}\alpha-\frac{2}{5}\left(\frac{\sigma-\alpha}{\sigma-1}\right)^{5/2}+\frac{2}{5}(1-\alpha)^{5/2}\left(\frac{\sigma}{\sigma-1}\right)^{5/2}\right]A \right. + \left[3\left(\frac{\sigma-\alpha}{\sigma-1}\right)^{1/2}\alpha-2\left(\frac{\sigma-\alpha}{\sigma-1}\right)^{3/2}+2(1-\alpha)^{3/2}\left(\frac{\sigma}{\sigma-1}\right)^{3/2}\right] \left.\times\left(\frac{N}{h}-\frac{1-\alpha}{\sigma-1}A\right)\right\}. \tag{12}$$

If, in the upper dry layer, there is a plug ($\tan\theta\le\breve{\mu}$), its velocity is equal to the velocity $u_\zeta$ at the top of the mixture. If the upper dry layer is sheared ($\tan\theta>\breve{\mu}$), from eqn. (1) and the fact that, in the dry layer, $\mu=\tan\theta$ and $I=-u'/z^{1/2}$, the velocity there is equal to

$$u=u_\zeta+\frac{2}{3}\frac{\tan\theta-\breve{\mu}}{\chi}\left(\zeta^{3/2}-z^{3/2}\right). \tag{13}$$

The quantity (1-$\alpha$)$u_{dry}$ is, then, equal to

$$(1-\alpha)u_{dry}=\begin{cases}(1-\alpha)u_\zeta & \text{if } \tan\theta\le\breve{\mu}\\ (1-\alpha)u_\zeta+\frac{2}{5}\frac{h^{3/2}}{\chi}(1-\alpha)^{5/2}(\tan\theta-\breve{\mu}) & \text{if } \tan\theta>\breve{\mu}\end{cases}, \tag{14}$$

where $u_\zeta$ can be obtained from eqn. (11) with $z$ = (1-$\alpha$)$h$. With this and eqns. (12) and (14), eqn. (10) may be written as

$$\frac{u_A}{h^{3/2}}=\lambda_1\tan\theta-\lambda_2, \tag{15}$$

where the coefficients $\lambda_1$ and $\lambda_2$ are functions of the type of fluid and granular material (through $\sigma$, $\chi$ $\breve{\mu}$ and $\hat{c}$) and the degree of saturation (through $\alpha$ and $\beta$); expressions for them are given in table 1. In uniform flows, the friction slope is equal to the tangent of the angle of inclination of the free surface. An expression for the friction slope, $j$, for the particles to be used also in non-uniform flows, can, therefore, be obtained from eqn. (15), with $j = \tan\theta$,

$$j = \frac{1}{\lambda_1}\frac{u_A}{h^{3/2}} + \frac{\lambda_2}{\lambda_1}. \tag{16}$$

Table 1: Values of the coefficients in the flow rule for the particles (eqn. (15)).

| | |
|---|---|
| $\lambda_1$ when $\tan\theta \le \breve{\mu}$ | $\frac{2}{15\chi\hat{c}\sigma^{1/2}(\sigma-1)^3}\Big\{\Big[(3\sigma-5+2\alpha)(\sigma-\alpha)^{3/2}-\sigma^{3/2}(3\sigma-5)(1-\alpha)^{5/2}\Big](\hat{c}\sigma+1-\hat{c})$ $+5\Big[(\sigma-3+2\alpha)(\sigma-\alpha)^{1/2}-\sigma^{1/2}(\sigma-3)(1-\alpha)^{3/2}\Big]\Big[(\sigma-1)(\beta-1)-\sigma(1-\alpha)\Big]\Big\}$ |
| $\lambda_1$ when $\tan\theta > \breve{\mu}$ | $\frac{2}{15\chi\hat{c}\sigma^{1/2}(\sigma-1)^3}\Big\{\Big[(3\sigma-5+2\alpha)(\sigma-\alpha)^{3/2}-\sigma^{3/2}(3\sigma-5)(1-\alpha)^{5/2}\Big](\hat{c}\sigma+1-\hat{c})$ $+5\Big[(\sigma-3+2\alpha)(\sigma-\alpha)^{1/2}-\sigma^{1/2}(\sigma-3)(1-\alpha)^{3/2}\Big]\Big[(\sigma-1)(\beta-1)-\sigma(1-\alpha)\Big]$ $+3\hat{c}\sigma^{1/2}(\sigma-1)^3(1-\alpha)^{5/2}\Big\}$ |
| $\lambda_2$ when $\tan\theta \le \breve{\mu}$ | $\frac{2}{15\chi\sigma^{1/2}(\sigma-1)^2}\Big[(3\sigma-5+2\alpha)(\sigma-\alpha)^{3/2}-\sigma^{3/2}(3\sigma-5)(1-\alpha)^{5/2}\Big]\breve{\mu}$ |
| $\lambda_2$ when $\tan\theta > \breve{\mu}$ | $\frac{2}{15\chi\sigma^{1/2}(\sigma-1)^2}\Big[(3\sigma-5+2\alpha)(\sigma-\alpha)^{3/2}-\sigma^{3/2}(3\sigma-5)(1-\alpha)^{5/2}$ $+3\sigma^{1/2}(\sigma-1)^2(1-\alpha)^{5/2}\Big]\breve{\mu}$ |

## 2.2 Fluid

The average fluid velocity along $H$ is equal to

$$U_A = \frac{\alpha(1-\hat{c})U_m + (\beta-1)U_{cm}}{\alpha(1-\hat{c})+\beta-1}, \tag{17}$$

where $U_m$ and $U_{cm}$ are the mean fluid velocities in the mixture and in the upper clear fluid layer, respectively. Berzi and Jenkins [1, 2] have shown that the calculated difference between the fluid and the particle velocity is rather small (however, this does not permit the neglect of the drag force in the momentum balances (3) and (6), given the high values of the drag coefficient $C$). We can, therefore, assume that $U_m \approx u_m$.

The mean fluid velocity in the upper clear fluid layer can be obtained from the integration of the distribution of the fluid velocity there; the latter comes from the integration of eqn. (7), with the distribution of the fluid shear stress provided by eqn. (3) when $c = 0$. Hence,

$$U_{cm}(\beta-1) = U_0(\beta-1) + \frac{2(\beta-1)^{3/2}}{5k} h^{1/2} (\tan\theta)^{1/2}, \tag{18}$$

where $U_0$ is the fluid velocity at the base of the upper clear fluid layer, which can be obtained from eqn. (11) with $z = 0$. With this and eqns. (12) and (18), eqn. (17) reads

$$\frac{U_A}{H^{3/2}} = \Lambda_1 \tan\theta - \Lambda_2 + \frac{\Lambda_3}{H}(\tan\theta)^{1/2}, \tag{19}$$

where $\Lambda_1$, $\Lambda_2$ and $\Lambda_3$ are functions of the type of fluid and granular material (through $\sigma$, $\chi$ $\breve{\mu}$ and $\hat{c}$), the mixing length (through $k$), and the degree of saturation (through $\alpha$ and $\beta$), and their expressions are given in table 2. Once again, in uniform flows, the friction slope is equal to the tangent of the angle of inclination of the free surface. An expression for the friction slope, $J$, for the fluid, to be used also in non-uniform flows, can, therefore, be obtained from eqn. (19), with $J = \tan\theta$,

$$J = \left\{ \frac{-\Lambda_3 + \left[\Lambda_3^2 + 4\Lambda_1\left(\Lambda_2 H^2 + U_A H^{1/2}\right)\right]^{1/2}}{2\Lambda_1 H} \right\}^2. \tag{20}$$

## 3 Discussion

We have simplified the theory proposed by Berzi and Jenkins [1–3] to obtain explicit relations between the tangent of the angle of inclination of the free

Table 2: Values of the coefficients in the flow rule for the fluid (eqn. (19)).

| | |
|---|---|
| $\Lambda_1$ | $\frac{2(1-\hat{c})}{15\xi^{3/2}\chi\hat{c}\sigma^{1/2}(\sigma-1)^3[\alpha(1-\hat{c})+\beta-1]}\{[(5\alpha\sigma-3\alpha-2\sigma)(\sigma-\alpha)^{3/2}+2\sigma^{5/2}(1-\alpha)^{5/2}$ $+5(\sigma-1)^{5/2}(\beta-1)](\hat{c}\sigma+1-\hat{c})$ $+5[(3\alpha\sigma-\alpha-2\sigma)(\sigma-\alpha)^{1/2}+2\sigma^{3/2}(1-\alpha)^{3/2}][(\sigma-1)(\beta-1)-\sigma(1-\alpha)]$ $+15(\beta-1)^2(\sigma-1)^{5/2}\}$ |
| $\Lambda_2$ | $\frac{2(1-\hat{c})}{15\xi^{3/2}\chi\sigma^{1/2}(\sigma-1)^2[\alpha(1-\hat{c})+\beta-1]}[(5\alpha\sigma-3\alpha-2\sigma)(\sigma-\alpha)^{3/2}+2\sigma^{5/2}(1-\alpha)^{5/2}+5(\sigma-1)^{5/2}($ |
| $\Lambda_3$ | $\frac{2(\beta-1)^{3/2}}{5\xi^{1/2}k[\alpha(1-\hat{c})+\beta-1]}$ |

surface, the depth and the average particle velocity and between the tangent of the angle of inclination of the free surface, the depth and the average fluid velocity. Those two relations are the flow rules for the particles and the fluid, respectively, if one interprets the tangent of the angle of inclination of the free surface as the friction slope.

The fact that the friction slope for the particles has a different expression from that for the fluid is crucial to the expression of the resistances in two-phase, depth-averaged, mathematical models of non-uniform flows (see, for example, the steady granular-fluid wave over a rigid bed analysed in [3]). Most previous works either treat the mixture as a single phase fluid [10–14] or, although aware of the differences between the two phases, focus solely on the particle motion resistance [15, 16].

Existing models for the motion resistance of debris flows can be basically grouped in the four categories described in the following (although there are examples of resistance formula obtained by combining the characteristics of two categories, e.g. see [17]); however, all of them suffer from major drawbacks with respect to the formulas presented here.

Takahashi [15] obtains an expression for the resistance of over-saturated debris flows, based on a modified version of the dilatant model for the particle shear stresses in the inertial regime described by Bagnold [18] using kinetic arguments. Certainly, the merits of Takahashi expression were his taking into account the dependence of the stress ratio on the particle concentration and his incorporation of the effects of the fluid turbulence. However, his theory was incomplete, because it did not deal with under-saturated debris flows and because he characterized the particles only through their density.

Some authors [16, 19] suggest the use of Coulomb's law to express the friction at the base of a debris flow. However, Coulomb's law cannot explain the experimentally observed dependence of the friction slope on the average velocity and the depth [20], given that it implies a constant stress ratio at the bed. In the theory of Berzi and Jenkins [1–3], the stress ratio at the bed depends on the local inertial parameter, i.e. the velocity gradient.

Many authors employ some kind of non-Newtonian rheology for modelling the debris flow resistance [10–14]. This approach implies that the debris flow can be approximated as a single-phase fluid. This, perhaps, applies when the solid phase is composed mainly of fine sediments (e.g. for mud flows, see [4] for more details) - that is, when the inertia of the particles is negligible with respect to the fluid viscous forces; but not when the content of large particles is relevant (as for stony debris flows, see [4]). The assumed non-Newtonian behaviour of the debris flow is, moreover, entirely phenomenological and, therefore, not well physically-based. Although the GDR MiDi rheology adopted here might also seem phenomenological, its physical link with the particle interactions at the micromechanical level has been demonstrated [21].

Finally, a few authors [14, 17] employ empirical expressions for the friction slope based on that for purely turbulent fluids (the Manning equation). These are not physically based and there are no rational arguments to justify their usage.

The formulas for the motion resistance of particles and fluid in debris flows proposed in the present work seem promising for practical application in the field of civil engineering.

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