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Optimal design of multibody systems

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Abstract.

Dynamic simulation is often used to predict the behaviour of multibody systems but, it should be sometimes completed to optimize the choice of the design parameters by taking into account the expected performances of the mechanism. The aim of this paper is to propose an optimal design method adapted to mechanisms containing close loops and submitted to dynamic criteria.

A formulation based on relative coordinates and Newton-Euler laws has been chosen, the constraint equations being expressed by the closing of the loops. A single value decomposition method is used to divide the set of differential and algebraic equations into two sub-spaces associated to the dependent and independent parameters. The equations of motion are integrated by the Newmark algorithm in its residual formulation.

The optimization step is performed by the steepest descent method with constraint compensation.

A special design sensitivity analysis has been developed by considering time dependent constraints, including first and second derivatives of the configuration parameters. The primary problem is reformulated so that integrals replace the time dependent functions. The classical adjoint variables are introduced to eliminate the state variables in the sensitivity formulation.

The method has been applied to optimize the suspensions of an urban railway vehicle.

1 Introduction.

When designing a multibody system, the dynamic analysis step is useful to predict the behaviour of the mechanism for a special set of *design variables*. These ones have to be optimized to improve the performances of the system without passing the conceptual or technological boundaries. Nevertheless when dynamic behaviour is taken into account, the optimization process becomes harder because both performances and constraints can explicitly depend of the times.

The purpose of this paper is to describe a general method for the optimization of the dynamic behaviour of multibody systems. A classical non linear

optimization approach has been adapted to suit to special features of mechanical systems. The dynamic analysis step is based on a relative coordinates approach that uses so-called *kinetostatic elements*.

The time simulation is connected to the optimization process by the classical use of the *adjoint variables*.

2 Mathematical preliminaries.

2.1 Optimality necessary and sufficient conditions.

Let $\underline{b}$ be the k design variables. The nonlinear programming problem (NLP) is usually defined as the minimization of a so-called *cost function* $\Psi_0(\underline{b})$ subject to n *equality constraints* and *(m-n) inequality constraints* $\Psi_i(\underline{b})$:

$$\begin{cases} \min \Psi_0(\underline{b}) \\ \Psi_i(\underline{b}) = 0 \quad i = 1,..,n \\ \Psi_i(\underline{b}) \le 0 \quad i = n+1,..,m \end{cases} \tag{1}$$

Let the cost and constraint functions be differentiable and $\underline{b}^* \in \Re^k$ be a local minimum for the NLP. The *Kuhn-Tucker necessary* condition expresses that there exists a multiplier vector $\underline{v} \in \Re^m$ such that [1] :

$$\begin{aligned} v_i &> 0 \quad i = 1,..,n \\ v_i &\ge 0 \quad i = n+1,..,m \\ v_i \Psi_i(\underline{b}^*) &= 0 \quad i = n+1,..,m \\ L(\underline{b},\underline{v}) &= \Psi_0(\underline{b}) + \underline{v}^T \underline{\Psi}(\underline{b}) \\ \frac{\partial L(\underline{b}^*,\underline{v})}{\partial b_j} &= 0 \quad j=1,..,k \end{aligned} \tag{2}$$

The Lagrange function $L(\underline{b},\underline{v})$ from cost and constraint functions is stationary at the optimum.

Figure 1 illustrates that condition for a *constraint set* of allowable solutions lays between two inequality constraints Ψ_1 and Ψ_2. The set of the allowable directions $\underline{y}$ that satisfy the constraints, represents a cone with $\underline{b}^*$ as apex and with tangents to the constraints at $\underline{b}^*$ as sides. A local minimum can be expected at this point if all allowable directions $\underline{y}$ form an obtuse angle with the direction of the non constrained minimum $-\underline{\nabla}\Psi_0$. In that case, the vector $-\underline{\nabla}\Psi_0$ may be written as a linear combination of $\underline{\nabla}\Psi_i$ with positive coefficients v_i :

$$-\underline{\nabla}\Psi_0 = \sum v_i \underline{\nabla}\Psi_i \rightarrow \underline{\nabla}L(\underline{b}^*,\underline{v}) = \underline{\nabla}\Psi_0 + \sum v_i \underline{\nabla}\Psi_i = 0 \tag{3}$$

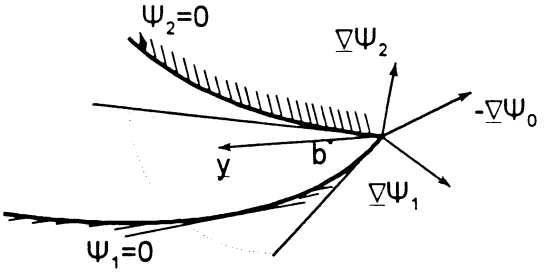

Figure 1 The Kuhn-Tucker condition.

The *sufficient* condition to let $\underline{b}^*$ become a local minimum is described in [1].

Often the designer wishes to know the benefit of relaxing constraints or the penalty associated with tightening them. One may consider the modified problem of minimizing $\Psi_o(\underline{b})$, subject to the constraints :

$$\begin{cases} \Psi_i(\underline{b}) = e_i & i = 1,..,n \\ \Psi_i(\underline{b}) \leq e_i & i = n+1,..,m \end{cases} \tag{4}$$

and evaluate the variation of $\Psi_0(\underline{b})$ versus $\underline{e}$. When both the necessary Kuhn-Tucker condition and the sufficient condition are verified, it may be shown [1] that, if v_i are positive for all the equality constraints, then the function $\underline{b}(\underline{e})$ is continuous and differentiable with the following property :

$$\frac{\partial \Psi_0(\underline{b}(\underline{e}))}{\partial e_i}\Big|_{\underline{e}=\underline{0}} = -v_i \qquad i = 1,..,m \tag{5}$$

The constraints with $v_i > 0$ decrease the cost function if they are relaxed, but the ones with $v_i < 0$ increase it. In order to minimize the cost function, the last constraints may be eliminated during the optimization process. This property yields to the choice of the correct descent direction especially when using a gradient projection method to solve the NLP problems.

2.2 Steepest descent method with constraint compensation.

The numerical optimization step comes from an iterative method that computes the variation $\delta\underline{b}$ that decreases the cost from an initial design estimation $\underline{b}_0$. In the design space, vector $\delta\underline{b}$ points to the direction of the minimum. Among all the allowable directions, the most efficient choice corresponds to the opposite side of the gradient of the cost function computed at the point $\underline{b}_0$. This yields :

$$\delta\underline{b} = -\alpha \underline{\nabla} \Psi_0^T(\underline{b}_0) \qquad \alpha > 0 \tag{6}$$

The direction of the steepest descent, given by the gradient of the cost function, is projected on the tangent to the constraints. The next iteration is chosen in such a

way that it decreases the cost without transgressing the constraints. Usually only the set of the tight inequality constraints (which are nearly zero) are taken into account. Hence the ϵ-*actives constraints* are defined by :

$$\begin{aligned} \Psi_i(\underline{b}) &= 0 \qquad\qquad i = 1,..,n \\ \Psi_i(\underline{b}) &\geq -\epsilon \quad \epsilon > 0 \quad i = n+1,..,m \end{aligned} \tag{7}$$

If we note the set of that kind of constraints $\underline{\tilde{\Psi}}(\underline{b})$, the problem becomes computing the variation $\delta\underline{b}$ that decreases $\Psi_0(\underline{b}_0 + \delta\underline{b})$ and satisfying $\underline{\tilde{\Psi}}(\underline{b})$. It may be written to the following local form :

searching : $$\delta\Psi_0 = \frac{\partial\Psi_0(\underline{b}_0)}{\partial\underline{b}}\delta\underline{b} = \underline{l}_0^T\delta\underline{b} \tag{8}$$

with the constraints : $$\delta\underline{\tilde{\Psi}} = \frac{\partial\underline{\tilde{\Psi}}(\underline{b}_0)}{\partial\underline{b}}\delta\underline{b} = \tilde{l}^T\delta\underline{b} \tag{9}$$

In order to annulate the violation of the constraints $\underline{\tilde{\Psi}}(\underline{b}_0)$, it is convenient to set $\delta\underline{\tilde{\Psi}} = -\underline{\tilde{\Psi}}(\underline{b}_0)$. Moreover $\delta\underline{b}$ should be small enough to avoid non linear effects. It is then useful to add the following further constraint to restrict the variation of $\delta\underline{b}$:

$$\delta\underline{b}^T W\delta\underline{b} \leq \xi^2 \tag{10}$$

where W is a diagonal weighting matrix.

Applying the necessary Kuhn-Tucker condition (2) to this problem, yields the *m* multipliers $\tilde{\mu}_i \geq 0$ $(i > n)$ and one multiplier $\gamma \geq 0$ such that [2] :

$$\begin{aligned} L(\underline{b},\tilde{\underline{\mu}},\gamma) &= \underline{l}_0^T\delta\underline{b} + \tilde{\underline{\mu}}^T\tilde{l}^T\delta\underline{b} + \gamma\,\delta\underline{b}^T W\delta\underline{b} \\ \frac{\partial L}{\partial b_j} = 0 \;\Rightarrow\; & \underline{l}_0 + \tilde{l}\tilde{\underline{\mu}} + 2\gamma W\delta\underline{b} = 0 \\ \tilde{\mu}_i(\delta\tilde{\Psi}_i + \tilde{\Psi}_i(\underline{b}_0)) &= 0 \quad i > n \\ \gamma(\delta\underline{b}^T W\delta\underline{b} - \xi^2) &= 0 \end{aligned} \tag{11}$$

The multipliers $\tilde{\mu}_i$ and $\gamma \geq 0$ should be positive. If one of them is negative, the associated constraint is eliminated from the set of ϵ-active constraints and a new computation of the multipliers is performed until all of them become strictly positive.

3 Optimization of multibody systems.

The optimization of the dynamic behaviour of multibody systems is complicated by the explicit time dependency. On one hand, the equations of motion, so-called *state equations*, are of course time dependent, as well as the cost and constraint functions. On the other hand, the configuration parameters $\underline{q}$ describing the motion of the multibody system, aren't design variables because their values are obtained from the integration of the state equations. They are called *state variables*.

Let first consider the case of a system described by N generalized coordinates without any kinematic constraints. Then the configuration parameters $\underline{q}$ fit the N degrees of freedom of the system.

The specific class of problems treated here includes minimization of extreme dynamic response, subject to performance constraints that must hold over the entire time interval $[0\ \tau]$. The cost function Ψ_0 that has to be minimized is written as :

$$\Psi_0 = \max_{t\in[0\ \tau]} f_0(\underline{b},\underline{q}(t),\dot{\underline{q}}(t),\ddot{\underline{q}}(t),t) \tag{12}$$

It can be for example the maximum acceleration of one of the bodies of a mechanism under operating conditions. The constraints are usually time dependent and express, for example, the limited range of the relative distance between two bodies during their motion :

$$\underline{\Phi}(\underline{b},\underline{q}(t),t) \le 0 \qquad t\in[0\ \tau] \tag{13}$$

Moreover the design variables $\underline{b}$, which concern geometric characteristics, inertia, stiffness or damping properties, have to stay between technological lower and upper boundaries $\underline{b}^L$ and $\underline{b}^U$ that lead to further constraints :

$$\underline{\Psi}(\underline{b}) \equiv \begin{cases} \underline{b}^L - \underline{b} \le \underline{0} \\ \underline{b} - \underline{b}^U \le \underline{0} \end{cases} \tag{14}$$

As it has been shown in 2.2 the optimization process needs the calculation of sensitivities versus the design variables, which are rather difficult to compute in presence of state variables and time dependent functions. The initial problem is therefore reformulated in the form of an equivalent functional formulation so that the constraints are transformed into time non-dependent functions. These are equal to zero when the constraints are satisfied and become positive otherwise [3]. Let define the "equivalent" operator <> such that :

$$\langle\phi(t)\rangle = \begin{cases} \phi(t) & si\ \phi(t) \ge 0 \\ 0 & si\ \phi(t) < 0 \end{cases} \tag{15}$$

the equivalent constraint during $[0\ \tau]$ is now replaced by :

$$\Psi = \int_0^\tau \langle\phi(t)\rangle dt \tag{16}$$

Because of the difficulty in treating the max-value of equation (12), a new design variable b_{k+1} is defined as the upper bound of f_0 and leads to the following inequality :

$$f_0(\underline{b},\underline{q},\dot{\underline{q}},\ddot{\underline{q}},t) - b_{k+1} \le 0 \qquad t\in[0\ \tau] \tag{17}$$

The cost function can now be taken as :

$$\Psi_0 = b_{k+1} \tag{18}$$

with the following further constraint :

$$\Psi_{m+1} = \int_0^{\tau} \langle f_0(\underline{b},\underline{z}(t),t) - b_{k+1} \rangle dt \qquad (19)$$

In practice, to adapt the optimization method to the dynamic behaviour of multibody systems, two steps have to be performed :

- firstly the dynamic analysis of the multibody system to get the time history of the state variables;
- secondly the calculation of the sensitivities of the time constraints versus the design variables.

4. Dynamic analysis.

4.1 Description of multibody systems.

The principle of a residual formulation is based upon the direct calculation of the *residues f* which physically correspond to the gap between the generalized forces that should be applied to get a given kinematic state and the generalized forces effectively applied by the actuators, springs or dampers.

The kinematic approach chosen to describe the motion of the system is based upon relative coordinates. The mechanical system is basically modelled as a set of open kinematic chains starting from the ground. In closed loop systems, the loops are first opened, imaging a joint to be cut, in order to obtain a tree-like mechanism that can be described as a set of kinematic chains. The cutting of a joint is taken into account by means of algebraic closure equations, corresponding to the geometric conditions imposed by the joint.

The kinematic chains are defined as a succession of frames, linked by so-called *kinetostatic elements* (figure 3), each of them being described by the relation between an entry frame (subscript e) and an exit frame (subscript s) and indifferently addresses joints and bodies. In more general way, a kinetostatic element can be seen as a movement and effort transformer (figure 2) [4].

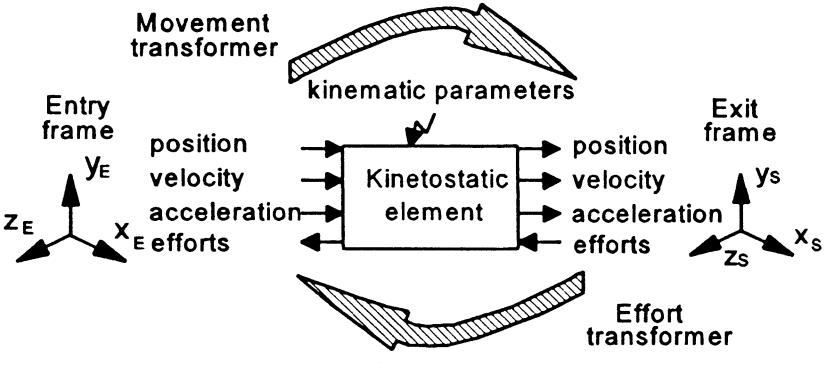

Figure 2 Principle of the kinetostatic element.

As a movement transformer, a kinetostatic element defines the relative situation of the exit and entry frames. Depending on whether there is a relative motion of the exit frame with respect to the entry frame, the element is said to be either active (e.g. articulation) or passive (e.g. rigid body). In the case of an active element, the relative motion is described in terms of certain so-called *kinematic parameters* associated with the element (e.g. the angle for a revolute articulation). It is then possible, for each element, to calculate the positions, velocities and accelerations of the exit frame, from the ones of the entry frame, and in terms of the values, first and second derivatives of the eventual kinematic parameters $\underline{q}$ associated with the element. For example the absolute rotation and translation velocities of the exit frame $\underline{\omega}_s$ and $\underline{v}_s$ are computed by the following formula :

$$\underline{v}_s = \underline{v}_e + \underline{\omega}_e \times \underline{r}(\underline{q})_{s/e} + \underline{v}(\underline{q},\dot{\underline{q}})_{s/e} \quad (20)$$

$$\underline{\omega}_s = \underline{\omega}_e + \underline{\omega}(\underline{q},\dot{\underline{q}})_{s/e} \quad (21)$$

where $\underline{\omega}_e$ and $\underline{v}_e$ are the absolute rotation and translation velocities of the entry frame and $\underline{r}_{s/e}$, $\underline{\omega}_{s/e}$ and $\underline{v}_{s/e}$ the relative positions, angular and linear velocities of the exit frame with respect to the entry frame, calculated from the kinematic parameters associated with the element.

As an effort transformer, a kinetostatic element defines how to compute the efforts at the entry frame from the ones at the exit frame, and the own contributions of the element, as :

$$\underline{F}_e = \underline{F}_s + \Delta\underline{F} \quad (22)$$

$$\underline{M}_e = \underline{M}_s + \underline{r}_{s/e} \times \underline{F}_s + \Delta\underline{M} \quad (23)$$

where $\underline{F}_e$, $\underline{M}_e$, $\underline{F}_s$ and $\underline{M}_s$ are the forces and moments applied to the frame *e* and *s* by the next elements and $\Delta\underline{F}$ and $\Delta\underline{M}$ are the internal contributions, coming especially from the inertia effects.

4.2 Equations of motion of an open loop system.

The residual formulation consists in a direct calculation of the residues relative to the configuration parameters, which correspond to the kinematic parameters associated with all active elements used to model the multibody system. It can be achieved by two successive recursions along the kinematic chains (figure 3) :

- a forward recursion, starting from the ground, in order to compute the kinematic state of all the frames, in terms of the values, first and second derivatives of the configuration parameters, by using the movement transformation relations (20,21);
- a backward recursion, starting from the end of the kinematic chains, in order to calculate the efforts in each frame, by the effort transformation relations (22,23).

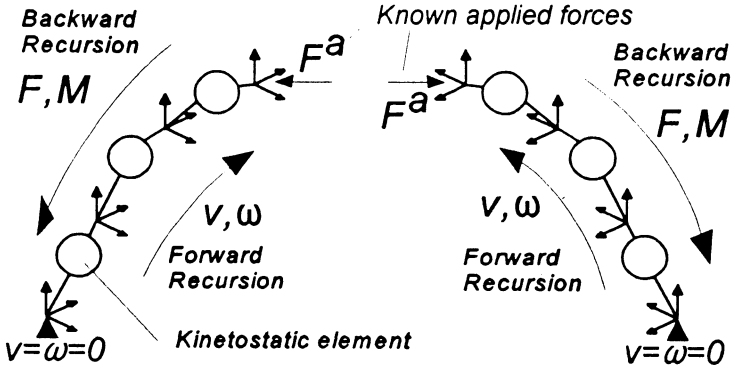

Figure 3 Forward and backward recursions.

The residue relative to a kinematic parameter associated with an active element can then be calculated by :

$$f_q = -\underline{F}_s \frac{\partial \underline{v}_{s/e}}{\partial \dot{q}} - \underline{M}_s \frac{\partial \underline{\omega}_{s/e}}{\partial \dot{q}} - \frac{\partial P_{int}}{\partial \dot{q}} \tag{24}$$

where P_{int} refers to the internal power of the element, induced by inertia forces and springs, dampers or actuators eventually associated with the element. The set of equations (there are as many equations as configuration parameters) yields to the system of differential equations where the unknown functions are the configuration parameters q and the first and second derivatives $\dot{q}$ and $\ddot{q}$.

$$\underline{f}(\ddot{q},\dot{q},q,t) = \underline{0} \tag{25}$$

4.3 Integration of the differential equations of motion.

Whatever the chosen method, integration is a step-by-step procedure which consists of determining from evaluations of the values, velocities and accelerations q^t, $\dot{q}^t$, $\ddot{q}^t$ at time t, the values, velocities and accelerations $q^{t+\Delta t}$, $\dot{q}^{t+\Delta t}$, $\ddot{q}^{t+\Delta t}$ at time $t+\Delta t$. The integration method is reorganized as a second-order scheme. In that case, the position and velocity of each degree of freedom at time $t+\Delta t$ can be expressed from the state at time t and the acceleration at time $t+\Delta t$ of the same degree of freedom, by means of integration formulas Δ and Δ' as [5] :

$$q^{t+\Delta t} = \Delta(q^t,\dot{q}^t,\ddot{q}^t,\ddot{q}^{t+\Delta t}) \tag{26}$$

$$\dot{q}^{t+\Delta t} = \Delta'(q^t,\dot{q}^t,\ddot{q}^t,\ddot{q}^{t+\Delta t}) \tag{27}$$

depending upon the chosen integration method and the time step. The set of the dynamic equations (25) can then be rewritten in order that the only unknowns are the accelerations $\ddot{q}^{t+\Delta t}$:

$$\underline{f}(q^t,\dot{q}^t,\ddot{q}^t,\ddot{q}^{t+\Delta t}) = 0 \rightarrow \underline{\mathcal{F}}(\ddot{q}^{t+\Delta t}) = 0 \tag{28}$$

Equation (28) is solved through the classical iterative Newton-Raphson method,

where the iteration $i+1$ is calculated from the preceding one, as :

$$\underline{q}_{i+1}^{t+\Delta t} = \underline{\ddot{q}}_{i}^{t+\Delta t} - J^{-1}\underline{\mathcal{F}}(\underline{\ddot{q}}_{i}^{t+\Delta t}) \tag{29}$$

where J refers to the iteration matrix of the system (28), defined by :

$$J_{ij} = \frac{\partial \mathcal{F}_i(\underline{\ddot{q}}^{t+\Delta t})}{\partial \ddot{q}_j^{t+\Delta t}} = \frac{\partial f_i}{\partial \ddot{q}_j^{t+\Delta t}} + \frac{\partial f_i}{\partial \dot{q}_j}\frac{\partial \Lambda_j'}{\partial \ddot{q}_j^{t+\Delta t}} + \frac{\partial f_i}{\partial q_j}\frac{\partial \Lambda_j}{\partial \ddot{q}_j^{t+\Delta t}} \tag{30}$$

$$J_{ij} = \frac{\partial \mathcal{F}_i(\underline{\ddot{q}}^{t+\Delta t})}{\partial \ddot{q}_j^{t+\Delta t}} = M_{ij} + CT_{ij}\frac{\partial \Lambda_j'}{\partial \ddot{q}_j^{t+\Delta t}} + KT_{ij}\frac{\partial \Lambda_j}{\partial \ddot{q}_j^{t+\Delta t}} \tag{31}$$

The matrices M, KT and CT are respectively called the *mass matrix* and the *stiffness* and *damping tangent matrices*.

In practice, a Newmark implicit integration scheme well adapted to second order differential equations has been chosen. In that case the integration formulas become :

$$\begin{aligned} \underline{q}^{t+\Delta t} &= \underline{q}^{t} + \Delta t\,\underline{\dot{q}}^{t} + (0.5-\beta)\Delta t^2\underline{\ddot{q}}^{t} + \beta\Delta t^2\underline{\ddot{q}}^{t+\Delta t} \quad 0<\beta<\tfrac{1}{2} \\ \underline{\dot{q}}^{t+\Delta t} &= \underline{\dot{q}}^{t} + (1-\gamma)\Delta t\,\underline{\ddot{q}}^{t} + \gamma\underline{\ddot{q}}^{t+\Delta t} \quad 0<\gamma<1 \end{aligned} \tag{32}$$

and the iteration matrix has the following form :

$$J = M + \gamma\Delta t CT + \beta\Delta t^2 KT \tag{33}$$

4.4 Kinematic constraint equations.

In presence of kinematic closed loops, the cutting of some joints provides a tree-like topology and induces a set of classical algebraic constraint equations :

$$\underline{b}(\underline{q}) = \underline{0} \tag{34}$$

These kinematic constraints produce constraint forces at the cut joints that must be taken into account in the equilibrium equations :

$$\underline{f}(\underline{\ddot{q}},\underline{\dot{q}},\underline{q},t) + \underline{f}_l = \underline{0} \tag{35}$$

It is well-known that the generalized constraint forces $\underline{f}_l$ are associated to the Jacobian matrix of the constraints by means of the Lagrange multipliers :

$$\underline{f}_l = B^T\underline{\sigma} \tag{36}$$

with :

- B : Jacobian matrix of the constraints $B_{ij} = \dfrac{\partial b_i(\underline{q})}{\partial q_j}$;
- $\underline{\sigma}$: vector of the Lagrange multipliers.

The dynamic equilibrium equations of the constrained mechanism are composed of differential equations (35) and algebraic equations (34) expressed for example in the acceleration space domain :

$$\underline{f}(\underline{\ddot{q}},\underline{\dot{q}},\underline{q},t) + B^T\underline{\sigma} = \underline{0} \tag{37}$$

$$B\ddot{q} + \dot{B}\dot{q} = \underline{0} \tag{38}$$

With the help of integration formulas (26) and (27) and in the same way that gives the residual equations (28), it is possible to transform the integration of the equations (37) and (38) to the solution of a non linear system where the unknowns are the accelerations at the time $t+\Delta t$ and the Lagrange multipliers :

$$\underline{\mathcal{F}}(\ddot{q}^{t+\Delta t}, \underline{\sigma}) = 0 \tag{39}$$

$$\underline{G}(\ddot{q}^{t+\Delta t}) = 0 \tag{40}$$

By using the Newton-Raphson method, we then get :

$$\begin{bmatrix} \underline{\mathcal{F}}(\ddot{q}_i^{t+\Delta t}, \underline{\sigma}_i) \\ \underline{G}(\ddot{q}_i^{t+\Delta t}) \end{bmatrix} + \begin{bmatrix} J & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \Delta\ddot{q}_{i+1}^{t+\Delta t} \\ \Delta\underline{\sigma}_{i+1} \end{bmatrix} = \underline{0} \tag{41}$$

For solving the equations (41) we use a reduction method to partition the set of the configuration parameters in dependent and independent variables. In that case, the system (41) is simplified as there isn't anymore constraint equation and because the number of variables has decreased. In practice we use the singular decomposition value to automatically get the best degrees of freedom even when the Jacobian matrix is ill-conditioned [6].

4.5 Reduction formulation based on the singular value decomposition.

Let B be a $(m \times l)$ matrix with rank $r \leq m$. The singular value decomposition theorem tells that it exists r non zero eigenvalues that satisfy the relation :

$$B^T B\underline{y} = \mu\underline{y} \tag{42}$$

Thanks to the eigenvectors associated to each non zero eigenvalues μ_i, the matrix B is transformed in such a way :

$$B = [U_1\ U_2] \begin{bmatrix} S_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} = \begin{bmatrix} U_1 S_{11} V_1^T \\ 0 \end{bmatrix} \tag{43}$$

S_{11} : diagonal matrix composed by the r non zero eigenvalues;
U_1, U_2, V_1, V_2 : orthogonal matrices composed by the eigenvectors.
Let be :

$$\dot{q} = [V_1 V_2] \begin{bmatrix} \dot{\underline{x}}_r \\ \dot{\underline{x}}_n \end{bmatrix} \tag{44}$$

The set of equations is then transformed to :

$$B\dot{q} = \underline{0} \quad \leftrightarrow \quad U_1 S_{11} \dot{\underline{x}}_r = \underline{0} \tag{45}$$

where $\dot{\underline{x}}_r$ is chosen to satisfy the constraints but $\dot{\underline{x}}_n$ has no effect on them and then can freely change. Let apply this property to the Jacobian matrix : from the equations of motion (41) we get :

$$\begin{cases} \left[\underline{\mathcal{F}}(\ddot{\underline{q}}_i^{t+\Delta t}, \underline{\sigma}_i)\right] + J\left[V_1 V_2\right]\begin{bmatrix} \Delta\ddot{\underline{x}}_{r,i+1}^{t+\Delta t} \\ \Delta\ddot{\underline{x}}_{n,i+1}^{t+\Delta t} \end{bmatrix} + \begin{bmatrix} V_1^T S_{11}^T U_1^T \\ 0 \end{bmatrix}\Delta\underline{\sigma}_{i+1} = \underline{0} & (46) \\ \left[\underline{G}(\ddot{\underline{q}}_i^{t+\Delta t})\right] + \begin{bmatrix} U_1 S_{11} \\ 0 \end{bmatrix}\begin{bmatrix} \Delta\ddot{\underline{x}}_{r,i+1}^{t+\Delta t} \\ \Delta\ddot{\underline{x}}_{n,i+1}^{t+\Delta t} \end{bmatrix} = \underline{0} & (47) \end{cases}$$

$\ddot{\underline{x}}_r$ has to satisfy the constraints and is obviously associated to the dependent parameters, whereas $\ddot{\underline{x}}_n$ is taken as the degrees of freedom of the mechanism. By multiplying equation (46) by the orthogonal matrix $\left[V_1 V_2\right]^T$, we finally get :

$$\begin{cases} V_1^T J V_1 \Delta\ddot{\underline{x}}_{r,i+1}^{t+\Delta t} + V_1^T J V_2 \Delta\ddot{\underline{x}}_{n,i+1}^{t+\Delta t} + S_{11}^T U_1^T \Delta\underline{\sigma}_{i+1} = -V_1^T \underline{\mathcal{F}}(\ddot{\underline{q}}_i^{t+\Delta t}, \underline{\sigma}_i) \\ V_2^T J V_1 \Delta\ddot{\underline{x}}_{r,i+1}^{t+\Delta} + V_2^T J V_2 \Delta\ddot{\underline{x}}_{n,i+1}^{t+\Delta t} = -V_2^T \underline{\mathcal{F}}(\ddot{\underline{q}}_i^{t+\Delta t}, \underline{\sigma}_i) \\ U_1 S_{11} \Delta\ddot{\underline{x}}_{r,i+1}^{t+\Delta t} = -\underline{G}(\ddot{\underline{q}}_i^{t+\Delta t}) \end{cases} \quad (48)$$

This system of differential equations is diagonal and can be solved in that particular way : the third system is firstly solved because it only depends from $\Delta\ddot{\underline{x}}_{r,i+1}^{t+\Delta t}$, afterwards the second one, where there are $\Delta\ddot{\underline{x}}_{r,i+1}^{t+\Delta t}$ and $\Delta\ddot{\underline{x}}_{n,i+1}^{t+\Delta t}$ together and finally the first one to get the Lagrange multipliers. This method is interesting for automatically taking to account the kinematic constraints despite the fact that it needs more numerical calculations. Moreover it easily eliminates the redundant constraints and gets well conditioned Jacobian matrix [6].

5 Sensitivity analysis by the adjoint variable method.

5.1 Open loop multibody systems.

Let assume first that the multibody system has no closed loop and that the constraints for the optimization don't involve any acceleration. Let $\underline{F}(\dot{\underline{z}}, \underline{z}, \underline{b}, t) = \underline{0}$ stand for the classical first order differential equations of motion ($\underline{z} = [\dot{\underline{q}}, \underline{q}]^T$) and let $\underline{\lambda}$ be arbitrary time dependent functions, called *adjoint variables* or *adjoint functions*, defined by the following identity [3] :

$$\int_0^\tau \underline{\lambda}^T \underline{F} dt = 0 \quad (49)$$

The first order development of this equation yields :

$$\int_0^\tau \underline{\lambda}^T \left(\frac{\partial \underline{F}}{\partial \dot{\underline{z}}}\delta\dot{\underline{z}} + \frac{\partial \underline{F}}{\partial \underline{z}}\delta\underline{z} + \frac{\partial \underline{F}}{\partial \underline{b}}\delta\underline{b}\right) dt = 0 \quad (50)$$

$$\int_0^\tau \underline{\lambda}^T \frac{\partial \underline{E}}{\partial \dot{\underline{z}}} \frac{d\delta \underline{z}}{dt} dt + \int_0^\tau \underline{\lambda}^T \left(\frac{\partial \underline{E}}{\partial \underline{z}} \delta \underline{z} + \frac{\partial \underline{E}}{\partial \underline{b}} \delta \underline{b} \right) dt = 0 \qquad (51)$$

By integrating the first term by parts, one obtains after rearranging the corresponding terms [4] :

$$\int_0^\tau \left(\frac{\partial \underline{E}^T}{\partial \dot{\underline{z}}} \dot{\underline{\lambda}} - \frac{\partial \underline{E}^T}{\partial \underline{z}} \underline{\lambda} + \frac{d\left(\frac{\partial \underline{E}^T}{\partial \dot{\underline{z}}}\right)}{dt} \underline{\lambda} \right) \delta \underline{z} \, dt = \int_0^\tau \underline{\lambda}^T \frac{\partial \underline{E}}{\partial \underline{b}} \delta \underline{b} \, dt \qquad (52)$$

Adjoint variables, whose number is the same as constraint functions, may be chosen in such a way that :

$$\begin{cases} \dfrac{\partial \underline{E}^T}{\partial \dot{\underline{z}}} \dot{\underline{\lambda}}_i - \dfrac{\partial \underline{E}^T}{\partial \underline{z}} \underline{\lambda}_i + \dfrac{d\left(\frac{\partial \underline{E}^T}{\partial \dot{\underline{z}}}\right)}{dt} \underline{\lambda}_i = \dfrac{\partial \langle \phi_i \rangle}{\partial \underline{z}} \qquad i = 1,..,m \\ \underline{\lambda}_i(\tau) = 0 \end{cases} \qquad (53)$$

The i^{th} constraint sensitivity with its integral formulation is given by :

$$\Psi_i + \delta \Psi_i = \int_0^\tau \left(\langle \phi_i \rangle + \frac{\partial \langle \phi_i \rangle}{\partial \underline{b}} \delta \underline{b} + \frac{\partial \langle \phi_i \rangle}{\partial \underline{z}} \delta \underline{z} \right) dt \qquad (54)$$

Thanks to the special choice of the adjoint variables, equation (54) can be rewritten in such a way that only design variables $\underline{b}$ are concerned :

$$\Psi_i + \delta \Psi_i = \int_0^\tau \left(\langle \phi_i \rangle + \frac{\partial \langle \phi_i \rangle}{\partial \underline{b}} \delta \underline{b} + \underline{\lambda}_i^T \frac{\partial \underline{E}}{\partial \underline{b}} \delta \underline{b} \right) dt \qquad (55)$$

The sensitivity vector $\underline{l}_i$ of the i^{th} constraint is then reduced to :

$$\delta \Psi_i = \underline{l}_i^T \delta \underline{b} = \int_0^\tau \left(\frac{\partial \langle \phi_i \rangle}{\partial \underline{b}} + \underline{\lambda}_i^T \frac{\partial \underline{E}}{\partial \underline{b}} \right) \delta \underline{b} \, dt \qquad (56)$$

5.2 Closed loop multibody systems.

When the mechanism contains closed loops, the configuration parameters $\underline{z}$ are not independent. The computation of derivatives of the cost and constraint functions is not direct. However the singular value decomposition of the Jacobian matrix of kinematic constraints is helpful to express the derivatives versus the degrees of freedom $\underline{y}_n = [\dot{\underline{x}}_n, \underline{x}_n]^T$ in place of the configuration parameters $\underline{z}$:

$$\begin{aligned} \frac{\partial \underline{E}}{\partial \underline{y}_n} &= \frac{\partial \underline{E}}{\partial \underline{z}} [V_2 \; V_2] \\ \frac{\partial \phi_i}{\partial \underline{y}_n} &= \frac{\partial \phi_i}{\partial \underline{z}} [V_2 \; V_2] \end{aligned} \qquad (57)$$

This way takes into account the kinematic constraints with an identical formulation

regarding adjoint variables and sensitivities. Finally the derivatives of the equations of motion versus degrees of freedom $\underline{y}_n$ are expressed by the mass matrix M, the stiffness matrix K and the damping matrix C :

$$\frac{\partial \underline{E}}{\partial \underline{\dot{y}}_n} = \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix}$$
$$\frac{\partial \underline{E}}{\partial \underline{y}_n} = \begin{bmatrix} C & K \\ -I & 0 \end{bmatrix} \tag{58}$$

5.3 Constraints with accelerations.

When the acceleration of the configuration parameters are involved in the $\Psi_i(\underline{b})$, this adjoint variable method has to be transformed. The equations of motion lead to the following expression for the sensitivity of the configuration parameters versus the design variables :

$$\frac{\delta \underline{\dot{z}}}{\delta \underline{b}} = -\left[\frac{\partial \underline{E}}{\partial \underline{\dot{z}}}\right]^{-1} \left(\frac{\partial \underline{E}}{\partial \underline{z}} \frac{\delta \underline{z}}{\delta \underline{b}} + \frac{\partial \underline{E}}{\partial \underline{b}}\right) \tag{59}$$

In that particular case, the i^{th} constraint is transformed to :

$$\delta \phi_i = \frac{\partial \phi_i}{\partial \underline{\dot{z}}} \delta \underline{\dot{z}} + \frac{\partial \phi_i}{\partial \underline{z}} \delta \underline{z} + \frac{\partial \phi_i}{\partial \underline{b}} \delta \underline{b} \tag{60}$$

$$\frac{d\phi_i}{d\underline{b}} \delta \underline{b} = \left(\frac{\partial \phi_i}{\partial \underline{\dot{z}}} \frac{\delta \underline{\dot{z}}}{\delta \underline{b}} + \frac{\partial \phi_i}{\partial \underline{z}} \frac{\delta \underline{z}}{\delta \underline{b}} + \frac{\partial \phi_i}{\partial \underline{b}}\right) \delta \_ \tag{61}$$

$$\frac{d\phi_i}{d\underline{b}} \delta \underline{b} = \left[\left(\frac{\partial \phi_i}{\partial \underline{z}} - \frac{\partial \phi_i}{\partial \underline{\dot{z}}} \frac{\partial \underline{E}^{-1}}{\partial \underline{\dot{z}}} \frac{\partial \underline{E}}{\partial \underline{z}}\right) \frac{\delta \underline{z}}{\delta \underline{b}} - \frac{\partial \phi_i}{\partial \underline{\dot{z}}} \frac{\partial \underline{E}^{-1}}{\partial \underline{\dot{z}}} \frac{\partial \underline{E}}{\partial \underline{b}} + \frac{\partial \phi_i}{\partial \underline{b}}\right] \delta \underline{b} \tag{62}$$

With equation (58) the previous one is rewritten to :

$$\frac{d\phi_i}{d\underline{b}} \delta \underline{b} = \left[\left(\frac{\partial \phi_i}{\partial \underline{z}} - \frac{\partial \phi_i}{\partial \underline{\dot{z}}} [M^{-1}C \;\; M^{-1}K]\right) \frac{\delta \underline{z}}{\delta \underline{b}} - \frac{\partial \phi_i}{\partial \underline{\dot{z}}} [M^{-1} \;\; 0] \frac{\partial \underline{E}}{\partial \underline{b}} + \frac{\partial \phi_i}{\partial \underline{b}}\right] \delta \underline{b} \tag{63}$$

In that case the i^{th} adjoint variable is defined by :

$$\begin{cases} \dfrac{\partial \underline{E}^T}{\partial \underline{\dot{z}}} \underline{\dot{\lambda}}_i - \dfrac{\partial \underline{E}^T}{\partial \underline{z}} \underline{\lambda}_i + \dfrac{d\left(\frac{\partial \underline{E}^T}{\partial \underline{\dot{z}}}\right)}{dt} \underline{\lambda}_i = \dfrac{\partial \langle \phi_i \rangle}{\partial \underline{z}} - \dfrac{\partial \langle \phi_i \rangle}{\partial \underline{\dot{z}}} [M^{-1}C \;\; M^{-1}K] \\ \underline{\lambda}_i(\tau) = 0 \end{cases} \tag{64}$$

and sensitivity vector $\underline{l}_i$ gets the following expression similar to equation (56) :

$$\delta \Psi_i = \underline{l}_i^T \delta \underline{b} = \int_0^\tau \left(\frac{\partial \langle \phi_i \rangle}{\partial \underline{b}} + \left(\underline{\lambda}_i^T - \frac{\partial \langle \phi_i \rangle}{\partial \underline{\dot{z}}} [M^{-1} \;\; 0]\right) \frac{\partial \underline{E}}{\partial \underline{b}}\right) \delta \underline{b} \, dt \tag{65}$$

6 Example.

When designing the suspension of a railway urban vehicle, one of the main criteria deals with the lateral behaviour of the tramway when it is running on narrow curves with small radius. In order to maximize the comfort when crossing such curves, the lateral acceleration has to be minimized by choosing efficient suspension properties. Figure 5 illustrates the case of a railway vehicle composed by two carbodies and two motor bogies and which has to be adapted to specific traffic constraints corresponding to low radius curves (14 m) at the constant speed of 3.7 m/s (13.3 km/s). Figure 4 shows the kinematic chains describing the topology of the vehicle : it is composed of 4 bodies, 7 revolute joints and 2 curve-sliding joints. In practice the problem is to choice the yaw properties of the secondary suspension between the carbodies and the bogies in such a manner that the maximum lateral acceleration reached by the front of the vehicle is minimum. The design variables are the yaw stiffness and the damping value of the front (b_1 and b_3 respectively) and the rear bogies (b_2 and b_4). There are two kinds of constraints, on one hand, to limit the range of variation of the design

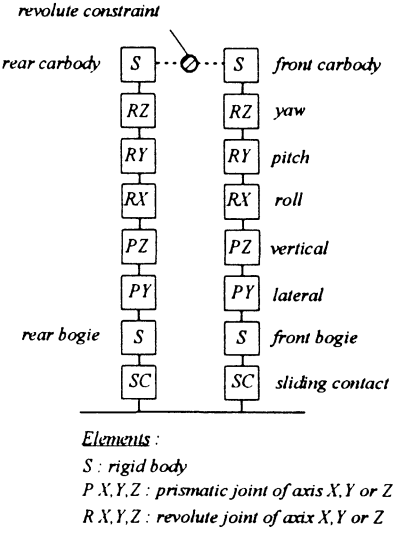

Figure 4 Topology of the vehicle.

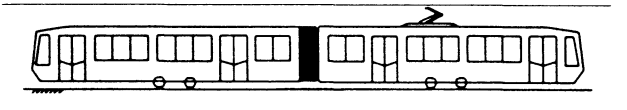

Figure 5 The Low Floor Vehicle.

variables and, on the other hand, to limit the relative yaw motion between carbodies and bogies.

Starting from initial design variables $\underline{b}_0$ = *(246 KNm/rad, 246 KNm/rad, 15 KNms/rad, 15 KNms/rad)*^{T}, the cost function is equal to 2.83 m/s². Figure 6 shows the evolution of the cost function during the iterative process and figure 7 illustrates the evolution of the design variables. The optimum is reached with $\underline{b}_{opt}$ = *(323 KNm/rad, 477 KNm/rad, 27 KNms/rad, 15 KNms/rad)*^{T} leading to a lower maximum lateral acceleration (1.91 m/s², figure 8) and satisfying all the constraints.

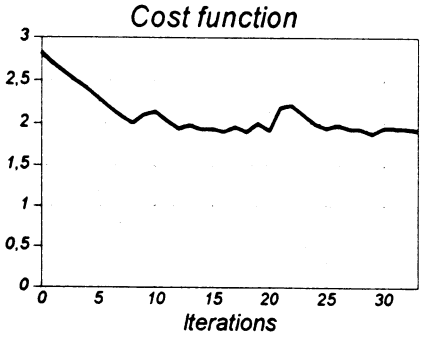

Figure 6 Evolution of the cost function during the optimization.

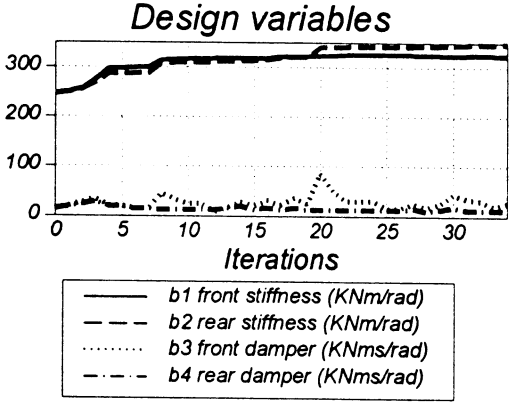

Figure 7 Evolution of the design variables during the optimization.

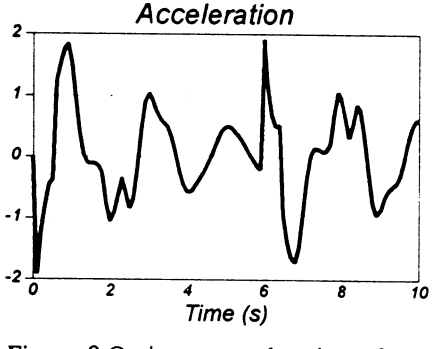

Figure 8 Optimum acceleration of the front vehicle.

7 Conclusions.

The optimization of the dynamic behaviour of multibody systems combines computer aided dynamic analysis of mechanisms and classical methods of optimization. A general method has been proposed, which uses formulation of dynamic equations based on kinetostatic elements and residual form integration; the optimization process is based on the steepest descent method with constraint compensation.

The equations of motion can be considered as constraint functions depending of state variables. It is useful to consider in the same time the state equations and the constraint functions by means of the classical adjoint variables to easily compute the sensitivity of the constraints and the cost function versus the design variables.

Because of the time dependency of the state equations, the constraints and the cost function are expressed into their equivalent integral form.

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