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Masonry-like materials with bounded compressive strength

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Abstract

This paper proposes a constitutive equation for masonry-like materials with bounded compressive strength. Successively a numerical method is proposed in order to solve the equilibrium problem of masonry-like solids with bounded compressive strength. In particular the derivative of the stress with respect to the total strain is calculated; this derivative will be used for calculating the tangent stiffness matrix and then for solving the non-linear system, obtained with the discretisation into finite elements via the Newton-Raphson method. Finally this numerical method is applied to the study of a three-dimensional circular reduced arch subjected to its own weight and a vertical load distributed along the extrados.

1 Introduction

Studying equilibrium problems of masonry solids requires above all a suitable choice for the constitutive equation of the material. Since these materials are in general very heterogeneous and their behaviour depends heavily on construction techniques, it appears extremely difficult to formulate constitutive equations general enough for application to all kinds of construction but yet simple enough to permit solution of the main boundary-value problems encountered in practice.

In several cases of masonry arches and vaults it seems quite realistic to suppose the material to be elastic, isotropic, non-resistant to traction and infinitely resistant to compression. More precisely, one supposes that the infinitesimal strain tensor $\mathbf{E}$ is the sum of an elastic part $\mathbf{E}^e$ and an inelastic part $\mathbf{E}^a$ and that the stress $\mathbf{T}$, negative semi-definite, depends linearly and isotropically on $\mathbf{E}^e$. Moreover, it is required that $\mathbf{E}^a$, interpreted as fracture strain, be positive semi-definite and orthogonal to $\mathbf{T}$. Thus one obtains a hyperelastic material, usually called masonry-like, which has been studied by many authors. In [1] and [2] the authors have proposed a numerical procedure for solving equilibrium problems of masonry structures via the finite element method. The application of this procedure to the study of arches [3], and vaults [4] subjected to concentrated loads shows that by using the constitutive equation of masonry-like materials, the mechanism of collapse and the corresponding multiplier can be determined very accurately.

The hypothesis that masonries are infinitely resistant to compression, although not conservative, is justified in many situations by the fact that for particular load conditions collapse occurs as a consequence of the mechanisms activated when the compression stress in the whole structure is inferior to its limit value. On the contrary, if an arch is subjected to its own weight and a vertical load distributed along the extrados, the line of thrust remains within the internal part of the arch and hinges do not form. Therefore, if the material were infinitely resistant to compression, the load could be increased indefinitely without ever reaching collapse. This result has suggested generalising the constitutive equation of masonry-like materials by introducing a bound to the compressive strength. More precisely, we suppose that, besides a positive semi-definite inelastic strain, a negative semi-definite inelastic strain may occur and be interpreted as crushing strain. Moreover, we suppose that the Cauchy stress has eigenvalues within the range $[-\sigma^c, \sigma^t]$, where σ^c and σ^t are two material constants called *compressive strength* and *tensile strength*, respectively. We obtain a hyperelastic whose definition is specified in Section 2.

In Section 3 we present the derivative of the stress with respect to the total strain. This is needed in order to calculate the tangent stiffness matrix used in solving equilibrium problems with finite elements via the Newton-Raphson method. The numerical method we obtain generalises the method presented in [1] and [2] for materials infinitely resistant to compression.

Finally, we apply the numerical method to the study of a three-dimensional circular reduced arch subjected to its own weight and a vertical load distributed along the extrados. The distributed load is progressively increased until collapse is reached; then, determination of the line of thrusts and the position of hinges at the instant of collapse allows easy interpretation of the results of the numerical analysis.

2 The constitutive equation

In this section we describe the constitutive equation of masonry-like materials with bounded compressive strength. Let us first present some notation: let $\mathcal{V}$ be a three-dimensional linear space and Lin the space of all linear applications of $\mathcal{V}$ into $\mathcal{V}$, equipped with the inner product $\mathbf{A}\cdot\mathbf{B} = \mathrm{tr}(\mathbf{A}^T\mathbf{B})$, $\mathbf{A}, \mathbf{B} \in \mathrm{Lin}$, with $\mathbf{A}^T$ the transpose of $\mathbf{A}$. Let us indicate as Sym, Sym^+ and Sym^- the subsets of Lin constituted by symmetric, symmetric positive semi-definite and symmetric negative semi-definite tensors, respectively.

Let us assume that the infinitesimal strain $\mathbf{E}$ is the sum of an elastic part $\mathbf{E}^e$ and of two mutually orthogonal inelastic parts $\mathbf{E}^t$ and $\mathbf{E}^c$ called *fracture strain* and *crushing strain*, respectively. $\mathbf{E}^t$ is positive semi-definite and $\mathbf{E}^c$ is negative semi-definite:

$$\mathbf{E} = \mathbf{E}^e + \mathbf{E}^t + \mathbf{E}^c, \tag{1}$$

$$\mathbf{E}^t \in \mathrm{Sym}^+, \qquad \mathbf{E}^c \in \mathrm{Sym}^-, \qquad \mathbf{E}^t\cdot\mathbf{E}^c = 0. \tag{2}$$

Moreover, we suppose that the Cauchy stress $\mathbf{T}$ depends linearly and isotropically on $\mathbf{E}^e$,

$$\mathbf{T} = 2\mu\mathbf{E}^e + \lambda\,\mathrm{tr}(\mathbf{E}^e)\mathbf{I}, \tag{3}$$

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where the Lamé moduli μ and λ of the material satisfy the inequalities

$$\mu > 0, \quad 2\mu + 3\lambda > 0. \tag{4}$$

Finally, we assume the existence of two positive material constants σ^t and σ^c, namely the tensile and compressive strength, respectively, such that

$$\mathbf{T} - \sigma^t \mathbf{I} \in \mathrm{Sym}^-, \quad \mathbf{T} + \sigma^c \mathbf{I} \in \mathrm{Sym}^+, \tag{5}$$

$$(\mathbf{T} - \sigma^t \mathbf{I}) \cdot \mathbf{E}^t = (\mathbf{T} + \sigma^c \mathbf{I}) \cdot \mathbf{E}^c = 0. \tag{6}$$

When σ^c goes to infinity, eqns (1)-(6) reduce to the constitutive equation of the masonry-like material infinitely resistant to compression studied in [1].
In [5] it is proven that, given $\mathbf{E} \in \mathrm{Sym}$, the Lamé moduli and the constants σ^c and σ^t, the constitutive eqns (1)-(6) has a unique solution $(\mathbf{T}, \mathbf{E}^t, \mathbf{E}^c)$. We denote by $\hat{\mathbf{T}} : \mathrm{Sym} \to \mathrm{Sym}$ the function which associates to every tensors $\mathbf{E}$ the stress $\mathbf{T} = \hat{\mathbf{T}}(\mathbf{E})$.

3 The numerical method

In this section we calculate the derivative $D_{\mathbf{E}}\hat{\mathbf{T}}$ of $\hat{\mathbf{T}}(\mathbf{E})$ with respect to $\mathbf{E}$. Knowing this derivative allows calculation of the tangent matrix and determination of the displacements by solving a non-linear system obtained by discretisation into finite elements via the Newton-Raphson method.
The algorithm used for the numerical solution of the equilibrium problem in the presence of incremental loads has already been described in [2] and it is thus omitted here.
Let us consider the orthonormal basis of Sym

$$\mathbf{O}_1 = \mathbf{q}_1 \otimes \mathbf{q}_1, \quad \mathbf{O}_2 = \mathbf{q}_2 \otimes \mathbf{q}_2, \quad \mathbf{O}_3 = \mathbf{q}_3 \otimes \mathbf{q}_3, \quad \mathbf{O}_4 = \frac{1}{\sqrt{2}}(\mathbf{q}_1 \otimes \mathbf{q}_2 + \mathbf{q}_2 \otimes \mathbf{q}_1),$$

$$\mathbf{O}_5 = \frac{1}{\sqrt{2}}(\mathbf{q}_1 \otimes \mathbf{q}_3 + \mathbf{q}_3 \otimes \mathbf{q}_1), \quad \mathbf{O}_6 = \frac{1}{\sqrt{2}}(\mathbf{q}_2 \otimes \mathbf{q}_3 + \mathbf{q}_3 \otimes \mathbf{q}_2), \tag{7}$$

where $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{q}_3$ are the eigenvectors of $\mathbf{E}$, the desired expression of $D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E})$ is

$$\mathbf{E} \in \mathcal{R}_1, \qquad D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E}) = 2\mu\, \mathbb{1} + \lambda\, \mathbf{I} \otimes \mathbf{I} \;;$$

$$\mathbf{E} \in \mathcal{R}_2, \qquad D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E}) = \frac{2\mu}{2+\alpha}\,\frac{\varepsilon^c + 2(1+\alpha)e_2 + \alpha e_3}{e_2 - e_1}\,\mathbf{O}_4 \otimes \mathbf{O}_4 +$$

$$+ \frac{2\mu}{2+\alpha}\,\frac{\varepsilon^c + 2(1+\alpha)e_3 + \alpha e_3}{e_3 - e_1}\,\mathbf{O}_5 \otimes \mathbf{O}_5 +$$

$$+ 2\mu\ \mathbf{O}_6 \otimes \mathbf{O}_6 + \frac{\mu(2+3\alpha)}{2+\alpha}\ (\mathbf{O}_2 + \mathbf{O}_3) \otimes (\mathbf{O}_2 + \mathbf{O}_3) +$$

$$+\ \mu\ (\mathbf{O}_2 - \mathbf{O}_3) \otimes (\mathbf{O}_2 - \mathbf{O}_3)\ ;$$

$$\mathbf{E} \in \mathcal{R}_3, \qquad D_E\hat{\mathbf{T}}(\mathbf{E}) = \frac{\mu}{1+\alpha}\ \frac{\varepsilon^c + (2+3\alpha)e_3}{e_3 - e_1}\ \mathbf{O}_5 \otimes \mathbf{O}_5 +$$

$$+ \frac{\mu}{1+\alpha}\ \frac{\varepsilon^c + (2+3\alpha)e_3}{e_3 - e_2}\ \mathbf{O}_6 \otimes \mathbf{O}_6 + E\ \ \mathbf{O}_3 \otimes \mathbf{O}_3\ ;$$

$$\mathbf{E} \in \mathcal{R}_4, \qquad D_E\hat{\mathbf{T}}(\mathbf{E}) = \mathbb{O}\ ;$$

$$\mathbf{E} \in \mathcal{R}_5, \qquad D_E\hat{\mathbf{T}}(\mathbf{E}) = \frac{2\mu}{2+\alpha}\ \frac{\varepsilon^t - 2(1+\alpha)e_1 - \alpha e_2}{e_3 - e_1}\ \mathbf{O}_5 \otimes \mathbf{O}_5 +$$

$$+ \frac{2\mu}{2+\alpha}\ \frac{\varepsilon^t - 2(1+\alpha)e_2 - \alpha e_1}{e_3 - e_2}\ \mathbf{O}_6 \otimes \mathbf{O}_6 +\ 2\mu\ \mathbf{O}_4 \otimes \mathbf{O}_4 +$$

$$+ \frac{\mu(2+3\alpha)}{2+\alpha}\ (\mathbf{O}_1 + \mathbf{O}_2) \otimes (\mathbf{O}_1 + \mathbf{O}_2) +$$

$$+\ \mu\ (\mathbf{O}_1 - \mathbf{O}_2) \otimes (\mathbf{O}_1 - \mathbf{O}_2)\ ;$$

$$\mathbf{E} \in \mathcal{R}_6, \qquad D_E\hat{\mathbf{T}}(\mathbf{E}) = \frac{\mu}{1+\alpha}\ \frac{\varepsilon^t - (2+3\alpha)e_1}{e_2 - e_1}\ \mathbf{O}_4 \otimes \mathbf{O}_4 +$$

$$+ \frac{\mu}{1+\alpha}\ \frac{\varepsilon^t - (2+3\alpha)e_1}{e_3 - e_1}\ \mathbf{O}_5 \otimes \mathbf{O}_5 + E\ \ \mathbf{O}_1 \otimes \mathbf{O}_1\ ;$$

$$\mathbf{E} \in \mathcal{R}_7, \qquad D_E\hat{\mathbf{T}}(\mathbf{E}) = \mathbb{O}\ ;$$

$$\mathbf{E} \in \mathcal{R}_8, \qquad D_E\hat{\mathbf{T}}(\mathbf{E}) = \frac{\sigma^t + \sigma^c}{e_2 - e_1}\ \mathbf{O}_4 \otimes \mathbf{O}_4 + \frac{\sigma^t + \sigma^c}{e_3 - e_1}\ \mathbf{O}_5 \otimes \mathbf{O}_5\ ;$$

$$\mathbf{E} \in \mathcal{R}_9, \qquad D_E\hat{\mathbf{T}}(\mathbf{E}) = \frac{\sigma^t + \sigma^c}{e_3 - e_1}\ \mathbf{O}_5 \otimes \mathbf{O}_5 + \frac{\sigma^t + \sigma^c}{e_3 - e_2}\ \mathbf{O}_6 \otimes \mathbf{O}_6\ ;$$

$$\mathbf{E} \in \mathcal{R}_{10}, \qquad D_E\hat{\mathbf{T}}(\mathbf{E}) = \frac{\mu}{2(1+\alpha)}\ \frac{\alpha\varepsilon^t + (2+\alpha)\varepsilon^c + 2(2+3\alpha)e_2}{e_2 - e_1}\ \mathbf{O}_4 \otimes \mathbf{O}_4 +$$

$$+ \frac{\mu}{2(1+\alpha)}\ \frac{\alpha\varepsilon^c + (2+\alpha)\varepsilon^t - 2(2+3\alpha)e_2}{e_3 - e_2}\ \mathbf{O}_6 \otimes \mathbf{O}_6 +$$

$$+ \frac{\sigma^t + \sigma^c}{e_3 - e_1}\ \mathbf{O}_5 \otimes \mathbf{O}_5 + \frac{\mu(2+3\alpha)}{1+\alpha}\ \mathbf{O}_2 \otimes \mathbf{O}_2,, \qquad (8)$$

where $\mathbb{1}$ and $\mathbb{O}$ are the fourth-order identity tensor and the fourth-order null tensor, $\alpha = \lambda/\mu$, $\varepsilon^c = \sigma^c/\mu$, $\varepsilon^t = \sigma^t/\mu$ and E is the Young modulus of the material. Moreover, e_1, e_2 and e_3 are the eigenvalues of **E**, ordered in such a way that $e_1 \le e_2 \le e_3$. Finally, $\Re_i$, i = 1, 10 are suitable regions of Sym [5].

4 A numerical example

Let us consider the reduced circular arch whose springings are fixed shown in Figure 1. The arch is subjected to its own weight and a load p, constant per unit span, distributed along the extrados. For symmetry reasons, only a quarter of the structure was studied and this was discretised into 300 isoparametric three-dimensional elements with 20 nodes and 27 Gauss points. We suppose that the material constituting the arch is not resistant to traction ($\sigma^t = 0$) and has a compressive strength $\sigma^c = 8.82$ MPa. The distributed load is progressively increased until the value p_c, beyond which the convergence cannot be reached; p_c, interpreted here as collapse load, resulted equal to 0.405 MPa.
Collapse occurs because of the formation of a number of hinges sufficient to render the structure labile. The constitutive characteristics of the material suggest supposing that at the instant of collapse in the normal sections of the arch where there are the hinges, the normal stress σ is constant and equal to σ^c in an interval having an extremum coinciding with the intrados or the extrados and nil elsewhere. Figure 2 where $\sigma = \sigma^c$ for $d \le y \le h/2$ and $\sigma = 0$ for $-h/2 \le y \le d$, shows one of these situations.

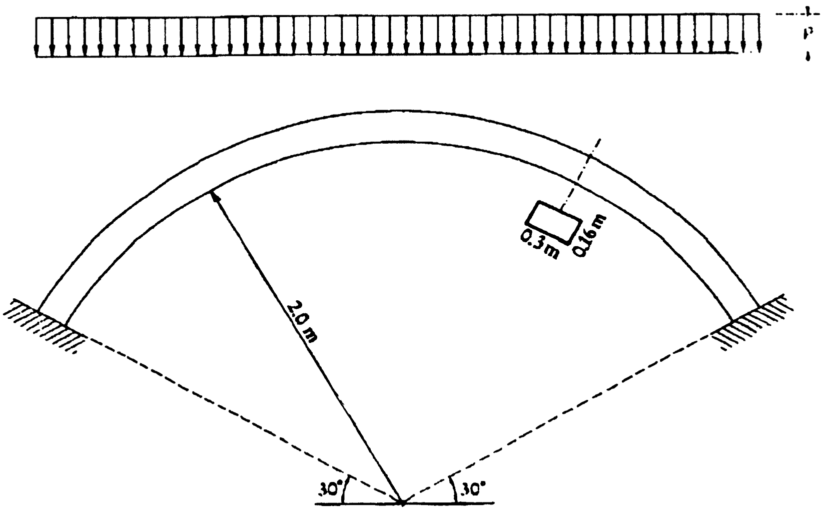

Figure 1. The reduced circular arch.

Let us suppose that a section of the arch is at the limit state. Let N, M and e = M/N be the normal force, bending moment and eccentricity, respectively; from Figure 2 we deduce

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$$N = 2\,\sigma^c \left(\frac{h}{2} - |e| \right), \tag{9}$$

from which, putting $N^c = h\sigma^c$, we obtain

$$d = -\frac{h}{2}\left(1 - \frac{2N}{N^c}\right)\frac{|M|}{M}, \quad |M| = -\frac{Nh}{2}\left(1 - \frac{N}{N^c}\right), \quad |M| = -\frac{Nh}{2}\left(1 - \frac{N}{N^c}\right). \tag{10}$$

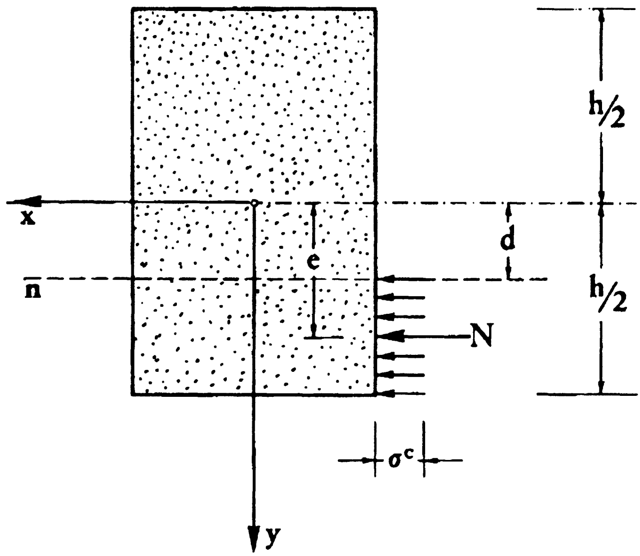

Figure 2 . Stress distribution in a normal section which is a hinge site.

Then let us consider any section, not necessarily at the limit state; from eqn $(10)_3$ we deduce $|e| \leq \frac{1}{2}h(1 - N/N^c)$, therefore, that for each plane parallel to the mean plane, the line of thrust is contained entirely within the region of the arch delimited by the two curves of *maximum eccentricity,* having equations $y = -\frac{1}{2}h(1 - N/N^c)$ and $y = \frac{1}{2}h(1 - N/N^c)$. For each normal section, let us put

$$\psi(N, M) = M + \frac{Nh}{2}\left(1 - \frac{N}{N^c}\right)\frac{|M|}{M}. \tag{11}$$

The *admissible region* , defined by the relation $\psi(N, M) \leq 0$, is constituted by all pairs $\mathbf{q} = (N, M)$ which are compatible with the constitutive properties of the material; the curve Γ having equation $\psi(N, M) = 0$ is called *limit curve* (Figure 3). In order for a section to be a hinge site, it is necessary that the relative normal force N and bending moment M belong to Γ; in this case the line of thrust is tangent to one of the two curves of maximum eccentricity.

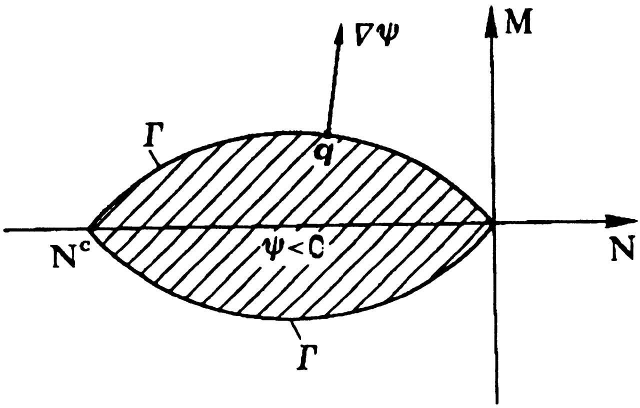

Figure 3. The admissible region.

Let us now suppose that $\mathbf{q}$ = (N, M) belongs to Γ and let us put α = (δ, φ), where δ is the extension of the mean line of the arch and φ is the relative rotation of the section. It is easily verified that from eqn (6) it follows that the generalised displacement α is *kinematically admissible* if and only if it has the same direction of the gradient of ψ calculated at $\mathbf{q}$. Thus, in view of eqns (11) and $(10)_1$ δ and φ must satisfy the condition $\delta = -\varphi d$. So, at the instant of collapse, the sections which are hinge sites rotate round the neutral axis.
In order to calculate the internal work $\mathcal{L}_i$ for each hinge, let us consider $\mathbf{q}$ belonging to Γ and α an admissible generalised displacement. From eqn $(10)_2$, recalling that $N^c = h\sigma^c < 0$ and by assuming φ and M to have the same sign, we immediately deduce

$$\mathcal{L}_i = - \frac{N^2}{2\sigma^c} |\varphi| \qquad (12)$$

Figure 4 shows both the line of thrust and the two curves of maximum eccentricity drawn in correspondence of the last load increment for which the convergence is reached. From these curves one immediately deduces the mechanism of collapse, and the angles θ_i corresponding to the five hinges may be estimated with a close approximation. The distance of the five hinges from the mean line of the arch is determined using eqn $(10)_1$, the normal force in the section being already known.
Figure 5 shows the circumferential stress. The maximum level of compression is reached at the extrados, near the crown and the springing, and at the intrados in the proximity of haunches; in the remaining zones the circumferential stress is nil and the material does no work.
In order to check the results of the analysis, we can determine an upper bound of the collapse load by using the hinge positions and the corresponding values

of the normal force obtained numerically. In fact, in view of the virtual work principle, we obtain that the value of the collapse load is 0.42 MPa, with a 3% margin of error.

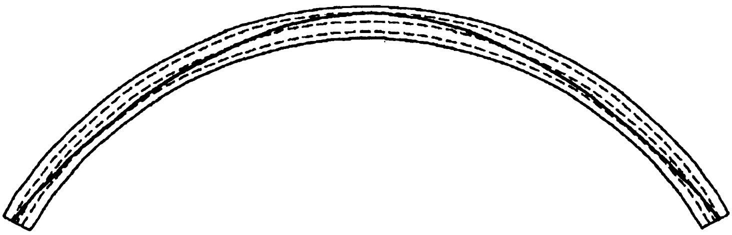

Figure 4. The line of thrust when collapse is reached.

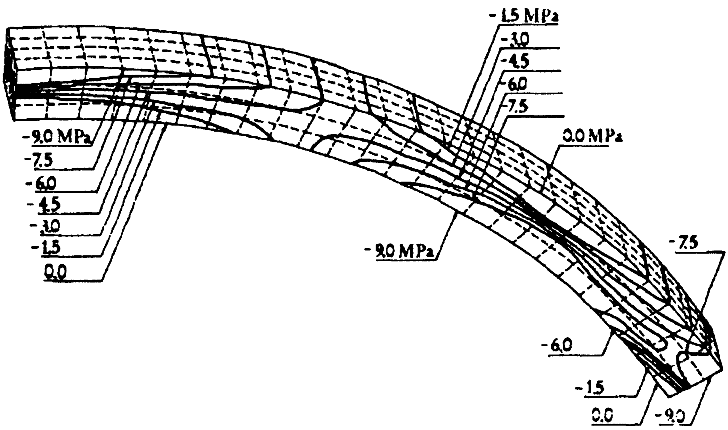

Figure 5. Circumferential stress distribution when collapse is reached.

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