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A Green's function method for detection of a cavity from one boundary measurement

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Abstract

A cavity, or a flux on the boundary of a cavity inside of a planar domain is to be determined from electrostatic or thermostatic measurements taken on the domain boundary. This inverse problem is solved by an optimization method, along with the Green's function modification of a boundary integral formulation of an elliptic partial differential equation which models voltage or heat conduction inside of the domain.

1 Introduction

Melnikov[6] has developed techniques which make it possible to obtain representations for Green's functions or matrices for elliptic partial differential equations with mixed boundary conditions, on domains of a variety of simple shapes. These Green's functions have found application to a number of problems in applied mechanics; to name a few, optimal shape design, contact mechanics with fixed or unknown contact zones, elastic and elasto–plastic torsion, heat conduction[6,7,8]. In this paper, Green's functions are used to solve inverse boundary value problems of internal cavity detection and flux reconstruction in planar domains.

In such inverse problems, a domain is occupied by an electrically or thermally conductive material which contains an internal flaw, perhaps an inclusion which has conductivity different from the surrounding material, or a cavity. From electrostatic or steady state thermal measurements at the boundary of the domain, we would like to characterize the internal flaw, or some other unknown property such as current flux across its boundary, under the assumption that voltage or heat satisfies Laplace's equation inside of the domain.

Several boundary element methods for solution of this type of problem have been proposed during the last decade. Notably, Bryan[2] derived an algorithm for thermal detection of a cavity and obtained convergence and continuous dependence results, as well. Boundary element methods have been applied to detection of cavities[4,11,14], crack detection problems[3,12] identification of unknown boundary conditions[10,11], and determination of regions of different conductivity[1,9]. Tanaka[13] gave a survey of boundary integral solutions for inverse problems with potential, elastostatic, and dynamic models. Several more examples are contained in the bibliographies of the above–mentioned references.

First, we reformulate the inverse problem as an integral equation on the boundary of the cavity. Then a least squares method is used to minimize the difference between data measured on the boundary and approximate data produced by the integral equation solution of the direct problem which contains the guessed cavity or flux. Previous boundary element methods for this type of problem used integral equations, but these equations were situated not only on the flaw boundary, but on the boundary of the larger domain, as well. Some advantages to using this Green's function method, then, are immediately obvious. Reduction in computation time occurs, especially during the optimization stage of solution, during which Laplace's boundary value problem must be solved repeatedly, at each iteration, in order to update approximate boundary data. Modification of grid spacing to treat domain boundaries with corners[2] is unnecessary. A Green's function method can solve inverse problems in domains with infinite boundaries. Not least, in a boundary integral equation formulation of the direct problem for Laplace's equation, the error is the difference between a single–layer potential and the true solution, so that the error is a harmonic function and has its maximum on the boundary of the domain; by eliminating the integral over the domain boundary, the maximum error is confined to the boundary of the guessed cavity; hence, simulated outer boundary data, which is used at each iteration as part of the minimization process, will contain less error than simulated data in boundary integral methods which utilize integrals over the outer boundary.

In section 1, the Green's function boundary integral method is given for solution of the elliptic partial differential equation (the direct problem); two inverse problems are formulated in section 2; in section 3, an algorithm is proposed for solution of the inverse problems; examples are presented in section 4; and numerical results are presented in section 5.

2 Direct Problem

Let Ω be a domain in the complex plane $\mathbf{C}$, with piecewise smooth boundary

$$\Gamma = \bigcup_{i=1}^{m} \Gamma_i$$

Suppose that an electrically or thermally conductive material occupies Ω, and that its conductivity is constant. The boundary value problem for voltage or temperature u in Ω is

$$\Delta u = -f \qquad z \in \Omega \tag{1}$$

$$\alpha_i \frac{\partial u}{\partial n_i} + \beta_i u = 0 \qquad z \in \Gamma_i, \quad i = 1, \dots m \tag{2}$$

where Δ is Laplace's operator, α_i, β_i are known functions not simultaneously zero on Γ_i, and n_i are unit normal vectors exterior to Γ_i; see Vekua[16] for theoretical details of this problem.

Let $G(z, \xi)$ be the Green's function which satisfies the boundary conditions (2) on Γ and

$$-\Delta_z G(z, \xi) = \delta(z - \xi), \qquad \xi, z \in \Omega; \tag{3}$$

that is, G has the property that as $z \to \xi$, $G(z, \xi) \to (-1/2\pi) \ln |z - \xi|$; then the solution u to (1), (2) may be written as

$$u(z) = \int_\Omega G(z, \xi) f(\xi) d\xi \qquad z \in \Omega \tag{4}$$

Now consider this related problem: that within Ω, there exists a hole D with boundary Γ_0, $\overline{D} \subset \Omega$; and

$$-\Delta u = f \qquad z \in \Omega \setminus D \tag{5}$$

$$\alpha_i \frac{\partial u}{\partial n_i} + \beta_i u = h_i \qquad z \in \Gamma_i, \quad i = 1, \dots, m \tag{6}$$

$$\alpha_0 \frac{\partial u}{\partial n_0} + \beta_0 u = g \qquad z \in \Gamma_0 \tag{7}$$

where possibly $g \equiv 0$.

In case g is not identically zero on Γ_0, but $f \equiv 0$ and all $h_i \equiv 0$, $i \geq 1$, the solution u can be constructed as follows. For $z \in \Omega \setminus D$, let

$$u(z) = \int_{\Gamma_0} G(z, \xi) \mu(\xi) d\Gamma_\xi \tag{8}$$

where G is the Green's function in (3), and take advantage of the logarithmic singularity of G by substitution of (8) into the well–known jump formula for the limit as $z \to \Gamma_0$ of the first normal derivative of a single layer potential[15] to obtain

$$\frac{\partial u}{\partial n_0} = \frac{1}{2}\mu(z) - \frac{\partial}{\partial n_0} \int_{\Gamma_0} G(z, \xi) \mu(\xi) d\Gamma_\xi, \quad z \in \Gamma_0$$

This formula, along with the boundary condition in (7), gives

$$\frac{1}{2}\alpha_0 \mu(z) - \int_{\Gamma_0} [\alpha_0 \frac{\partial G}{\partial n_0}(z, \xi) + \beta_0 G(z, \xi)] \mu(\xi) d\Gamma_\xi = g(z), \qquad z \in \Gamma_0 \tag{9}$$

where the normal derivative is with respect to z. This is a Fredholm integral equation for the density $\mu(z)$, $z \in \Gamma_0$. In general, this equation is ill–posed, but it is not hard to imagine conditions on α_0, β_0, e.g. $\alpha_0 = 1$, $\beta_0 = 0$, that make (9) an equation of the second kind, for which it is well known that a unique solution exists. Once this equation is solved for μ on Γ_0, (8) gives the solution u.

In the second case, g may or may not be identically zero, with nonhomogeneous boundary conditions $h_i \neq 0$, $i \geq 1$, We obtain, if possible, a twice differentiable function h for $z \in \Omega$ which satisfies the mixed conditions (6) on Γ. Then, considering only the special case $\alpha_0 = 1$, $\beta_0 = 0$, let $v = u - h$, and solve (5) for v in $\Omega \setminus D$ with $f(z) = -\Delta h(z)$, and homogeneous boundary conditions, except for where

$$\frac{\partial v}{\partial n_0}|_{\Gamma_0} = g - \frac{\partial h}{\partial n_0}|_{\Gamma_0} \tag{10}$$

In this case, multiplication of (5) by G, integration of both sides with respect to ξ, and integration by parts give the solution

$$v(z) = \int_{\Gamma_0} \frac{\partial G}{\partial n_0}(z,\xi)v(\xi)d\Gamma_\xi + \int_{\Gamma_0} G(z,\xi)[\frac{\partial h}{\partial n_0}(\xi) - g(\xi)]d\Gamma_\xi + \int_{\Omega\setminus D} G(z,\xi)f(\xi)d\xi \tag{11}$$

Now, if $z \to \Gamma_0$, then the jump formula can be used to obtain a Fredholm equation of the second kind for v on Γ_0

$$v(z) - 2\int_{\Gamma_0} \frac{\partial G}{\partial n_0}(z,\xi)v(\xi)d\Gamma_\xi = 2\int_{\Gamma_0} G(z,\xi)[\frac{\partial h}{\partial n_0}(\xi) - g(\xi)]d\Gamma_\xi + 2\int_{\Omega\setminus D} G(z,\xi)f(\xi)d\xi \qquad z \in \Gamma_0 \tag{12}$$

where the right hand side consists of known functions. Once solved for v on Γ_0, the formula (11) yields v for $z \notin \Gamma_0$. Then, $u = v + h$. Note that the two dimensional integrals (11), (12) can be converted by means of integration by parts to boundary integrals, since $f = \nabla h$.

3 Two Inverse Problems

The first inverse problem: Suppose Ω contains a cavity D with boundary Γ_0, the location, size, and shape of which are unknown and inaccessible to measurement. Assume that g is a known function on Γ_0. Then, given α_0, β_0, and assuming that (5), (6), (7) hold, determine D from one additional measurement $M_i[u] = s_i$ on some accesible part Γ_i of the boundary Γ. The s_i could be, for example, flux, or voltage or temperature.

The second inverse problem: Suppose that the location, size, and shape of the cavity D are known, and that α_0, β_0 are given, but the function g on Γ_0

is undetermined. Assuming (5), (6), (7), plus one additional measurement $M_i[u] = s_i$ for some i, taken on part of Γ, determine g.

Note that Γ_0 and g cannot be determined simultaneously from a single measurement pair (h_i, s_i), on Γ_i, for any or all $i = 1, \ldots, m$. An additional pair of experimental measurements on Γ would be required. To see this, let $\xi^* \in \Omega$. For any hole D in Ω which contains ξ^*, let g on Γ_0, the boundary of D, be given by $g(z) = \alpha_0 \partial G(z, \xi^*)/\partial n_0 + \beta_0 G(z, \xi^*)$. $G(z, \xi^*)$ satisfies (5) with $f \equiv 0$ for $z \notin D$. Assuming that D is uncharacterized, then so will be g on its boundary; yet data on Γ will be identical for all D which contain ξ^*.

4 Method of Solution

Here, an iterative method is proposed for numerical solution of the inverse problems, in which successive approximations are improved by means of a least–squares method[5].

In the case $f \equiv 0$ and data $h_i \equiv 0$ for all $i = 1, \ldots, m$, the algorithm begins as follows: 1) Collect measurement data s_i on Γ_i for some of the $i = 1, \ldots, m$. 2) Make an initial guess $\tilde{D}$ with boundary $\tilde{\Gamma}_0$ for unknown D in the first algorithm; or $\tilde{g}$ for unknown g in the second inverse problem. 3) Substitute this initial guess in formula (9), solve the resulting integral equation for $\tilde{\mu}$, and use (8) to obtain simulated measurement $\tilde{s}_i$ on the Γ_i.

Now consider the functional

$$F(\tilde{s}_i(\mathbf{c}), s_i) = \int_{\Gamma_i} |\tilde{s}_i(\xi; \mathbf{c}) - s_i(\xi)|^2 d\Gamma_\xi$$

which, for practical purposes, is a summation determined by, say, the trapezoid rule for numerical integration. Here, $\mathbf{c}$ is a set of parameters which characterize $\tilde{\Gamma}_0$ in the first inverse problem, $\tilde{g}$ in the second. We wish to find $\tilde{D}$ or $\tilde{g}$ which minimizes the difference F of simulated measurement $\tilde{s}_i$ and measurement s_i. In inverse problem 1, a parameterization which characterizes the guess $\tilde{\Gamma}_0$ must be available, of form $\tilde{\xi}(\tau) = \tilde{\xi}(\tau; \mathbf{c})$, where $0 \leq \tau \leq a$. In inverse problem 2, $\tilde{g}(\tau) = \tilde{g}(\tau; \mathbf{c})$, where ξ is given.

Then, assuming that $\mathbf{c} + \delta$, with δ to be determined, gives the closest parameterization of the unknown, consider the first order Taylor approximation of F,

$$\Phi(\mathbf{c} + \delta) = \int_{\Gamma_i} [\tilde{s}_i(\xi; \mathbf{c}) + \sum_{j=1}^{n} \frac{\partial \tilde{s}_i}{\partial c_j}(\xi; \mathbf{c})\delta_j - s_i(\xi)]^2 d\Gamma_\xi$$

where

$$\frac{\partial \Phi}{\partial \delta_k}(\mathbf{c} + \delta) = 2\int_{\Gamma_i} [\tilde{s}_i(\xi; \mathbf{c}) + \sum_{j=1}^{n} \frac{\partial \tilde{s}_i}{\partial c_j}(\xi; \mathbf{c})\delta_j - s_i(\xi)] \frac{\partial \tilde{s}_i}{\partial c_k}(\xi; \mathbf{c}) d\Gamma_\xi = 0$$

for $k = 1, \ldots, n$ yields the matrix equation

$$\mathbf{A}\delta = \mathbf{q}$$

where the $n \times n$ matrix $\mathbf{A} = (a_{kj})$ and $\mathbf{q} = (q_1, \ldots, q_n)$ have entries

$$a_{kj} = \int_{\Gamma_i} \frac{\partial \tilde{s}_i}{\partial c_j}(\xi; \mathbf{c}) \frac{\partial \tilde{s}_i}{\partial c_j}(\xi; \mathbf{c}) d\Gamma_\xi$$

and

$$q_k = \int_{\Gamma_i} (\tilde{s}_i(\xi; \mathbf{c}) - s_i(\xi)) \frac{\partial \tilde{s}_i}{\partial c_j}(\xi; \mathbf{c}) d\Gamma_\xi$$

respectively. Note that $\mathbf{A}$ already has square form $\mathbf{A} = \mathbf{P}^T\mathbf{P}$ suitable for application of a corrected Gauss–Newton scheme[5].

The derivatives $\partial \tilde{s}_i / \partial c_j$, for $z \in \Gamma_i$, are calculated on the parameterization of the integral (8), with $\tilde{\xi} = \xi$ in case g is to determined,

$$\tilde{u}(z) = \int_0^a G(z, \tilde{\xi}(\tau)) \tilde{\mu}(\tau) |\tilde{\xi}'(\tau)| d\tau$$

as follows:

$$\frac{\partial \tilde{s}_i}{\partial c_j}(z) = M_i[\frac{\partial \tilde{u}}{\partial c_j}](z) = \int_0^a M_i[G(z, \tilde{\xi}(\tau))] \frac{\partial \tilde{\mu}}{\partial c_j}(\tau) |\tilde{\xi}'(\tau)| d\tau +$$

$$\int_0^a M_i[\frac{\partial G}{\partial \tilde{\xi}}(z, \tilde{\xi}(\tau))] \frac{\partial \tilde{\xi}}{\partial c_j}(\tau) \tilde{\mu}(\tau) |\tilde{\xi}'(\tau)| d\tau + \int_0^a M_i[G(z, \tilde{\xi}(\tau))] \tilde{\mu}(\tau) \frac{\partial}{\partial c_j} |\tilde{\xi}(\tau)| d\tau$$

where the second two integral terms are zero in case Γ_0 is known and g is to be determined.

The derivatives $\partial \mu / \partial c_j$ are calculated from integral equations derived from (9), for $0 \leq \theta \leq a$

$$\frac{1}{2} \alpha_0 \frac{\partial \tilde{\mu}}{\partial c_j}(\theta) + \int_0^a [\alpha_0 \frac{\partial G}{\partial n_0}(\tilde{z}(\theta), \tilde{\xi}(\tau)) + \beta_0 G(\tilde{z}(\theta), \tilde{\xi}(\tau))] \frac{\partial \tilde{\mu}}{\partial c_j}(\tau) |\tilde{\xi}'(\tau)| d\tau$$
$$= - \int_0^a [\alpha_0 \frac{\partial G}{\partial n_0}(\tilde{z}(\theta), \tilde{\xi}(\tau) + \beta_0 G(\tilde{z}(\theta), \tilde{\xi}(\tau)] \tilde{\mu}(\tau) |\tilde{\xi}'(\tau)| d\tau + \frac{\partial \tilde{g}}{\partial c_j}(\theta)$$

where the $\partial \tilde{g} / \partial c_j$ term are zero if D is to be determined; the integral term on the right is zero if D is known and g is to be determined.

Hence, the next step 4) is to solve for δ by means of the corrected Gauss–Newton scheme[5]. Then 5) update $\mathbf{c}$ by letting $\mathbf{c} = \mathbf{c} + \delta$. The procedure is repeated until a convergence criterion is met, say $||\nabla_{\mathbf{c}} F|| \approx 0$. The final output $\mathbf{c}$ should parameterize the approximate solution to the inverse problem.

In case $g \equiv 0$ on Γ_0, or is near zero, hence too weak to produce useful data s_i, we must assume that on some Γ_i, a stimulus $h_j \neq 0$ is applied for some $j = 1, \ldots, m$, and a corresponding response s_i is measured on a Γ_i. In this case, we assume that $\alpha_0 = 1$, $\beta_0 = 0$, so that g represents flux across Γ_0, and use the formulae (11), (12) instead of (8), (9), to simulate data, and to obtain the derivatives of simulated data. The two dimensional integrals in (11), (12) must be converted to boundary integrals, before differentiation with respec to parameters.

5 Examples

1. Let Ω be the right circular sector of radius R, $\Omega = \{z = re^{i\theta} : 0 \leq r \leq R, 0 \leq \theta \leq \pi/2\}$. For the boundary Γ, Γ_1 is the circular part, Γ_2 is the interval $[0, R]$ on the x-axis, Γ_3 is the interval $[0, R]$ on the y-axis. The Green's function (3) for Ω with mixed boundary conditions

$$\frac{\partial u}{\partial r}|_{\Gamma_1} = u|_{\Gamma_2} = \frac{\partial u}{\partial \theta}|_{\Gamma_3} = 0 \tag{13}$$

is

$$G(z,\xi) = \frac{1}{2}\ln\frac{|z-\bar{\xi}||z+\xi||R^2 - z\xi||R^2 + z\bar{\xi}|}{|z-\xi||z+\bar{\xi}||R^2 - z\bar{\xi}||R^2 + z\xi|}$$

For the inverse problems 1 and 2 in Ω, we suppose that Γ_1 and Γ_3 are insulated, Γ_2 is grounded, i.e., (13) holds, and flux $\partial u/\partial n_0 = g$ on Γ_0. Then voltage or heat $u = s_3$ is measured on Γ_3, and the method of section 3 is applied to determine D or g. In inverse problem 1, if $g \equiv 0$, then we assume Γ_1 and Γ_3 are insulated and apply a voltage or heat $u = h_2 \neq 0$ to Γ_2, so that

$$\frac{\partial u}{\partial r}|_{\Gamma_1} = \frac{\partial u}{\partial \theta}|_{\Gamma_3} = 0, \quad u|_{\Gamma_2} = h_2 \tag{14}$$

where h_2 is smooth and has compact support in Γ_2. Then voltage or heat $u = s_3$ is measured on Γ_3. In order to apply formulae (11), (12) to this problem, let $h(r,\theta) = h_2(r)\cos 2\theta$, so that h satisfies (14) on Γ.

2. Ω is an infinite strip, $\Omega = \{z : 0 \leq \text{Im}z \leq \pi/2\}$. Γ_1 is the line $\text{Im}z = 0$, Γ_2 is the line $\text{Im}z = \pi/2$. Let

$$E_1(p) = |e^p + 1|, \qquad E_2(p) = |e^p - 1| \tag{15}$$

Then the Green's function for Ω with boundary conditions

$$u(x,0) = \frac{\partial u}{\partial y}(x, \frac{\pi}{2}) = 0, \quad u(\pm\infty, y) < \infty \tag{16}$$

is

$$G(z,\xi) = \frac{1}{2\pi}\ln\frac{E_1(z-\xi)E_2(z-\bar{\xi})}{E_2(z-\xi)E_1(z-\bar{\xi})} \tag{17}$$

For the inverse problems, suppose (16) is satisfied, i.e., the boundary Γ is grounded on Γ_1 and insulated on Γ_2, measure voltage or heat $u = s_2$

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on part of Γ_2, and apply the method in section 3. If $g \equiv 0$ on Γ_0, then keep Γ_2 insulated and prescribe $u = h_1$ on Γ_1, where h_1 is smooth with compact support, and measure $u = s_2$ on Γ_2. For application of (11), (12), let $h(z) = h_1(x)\cos(y)$

3. Let $\Omega = \{z : 0 \leq \mathrm{Re}z, 0 \leq \mathrm{Im}z \leq \pi\}$ be a semistrip, with Γ_1, Γ_2, Γ_3 the lower, left, and upper boundaries, respectively. Prescribe

$$\frac{\partial u}{\partial x}(0,y) = u(x,0) = u(\pi,0) = 0, \qquad u(+\infty,y) < \infty \tag{18}$$

on Γ. Let $E = E_2$ from (15). Then the Green's function for Ω with boundary conditions (17) is[6]

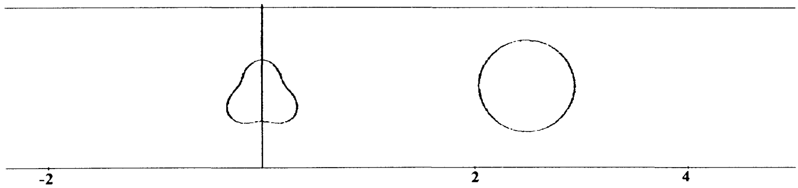
$$G(z,\xi) = \frac{1}{2\pi}\ln\frac{E(z+\bar{\xi})E(z-\bar{\xi})}{E(z-\xi)E(z+\xi)} \tag{19}$$

6 Numerical Results

Numerical implementation has begun in the cases of examples 1 and 2. For example, in the case of the infinite strip, we have the following result.

The unknown domain D is nonconvex and depends on four parameters, with boundary Γ_0 given by $\xi(\tau;\mathbf{c}) = c_1 + ic_2 + (1 - c_3 sin3\tau)c_4e^{i\tau}$, $0 \leq \tau \leq 2\pi$; see figure 1, below. Flux on Γ_0 is given as $g(\tau) = 2\sin(\tau/2)$, and data is collected at twenty-five points equally spaced on the interval $[0,4] \times \{\pi/2\}$. In figure 1, the unknown domain has parameters $\mathbf{c}_0 = (0, .7, .2, .3)$, and the initial guess is $\mathbf{c} = (2.5, .8, 0, .45)$. An iteration that exceeds the strip boundary is placed back inside of the strip; similarly, c_3 is constrained to fall between 0 and .3. In this case, the Euclidean norm $\|\mathbf{c} - \mathbf{c}_0\|$ converged in sixteen iterations to less that 10^{-6} units. Future plans include finding the limitations of the algorithm under conditions such as noisy data.

Figure 1:



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