Transactions on Modelling and Simulation vol 9, © 1995 WIT Press, www.witpress.com, ISSN 1743-355X

An alternative boundary element formulation for plate bending analysis

J.B. Paiva, L.O. Neto
Structures Department, São Carlos Civil Engineering School, 13560-250 São Carlos, SP, Brazil

Abstract

This work presents an alternative formulation for plate bending analysis in which the boundary equation of displacements and its derivative in directions normal and tangential to the boundary are used. This formulation considers three nodal values in displacements: w, $\partial w/\partial n$ and $\partial w/\partial t$ for each boundary node, but according to Kirchhoff's hypothesis, only two nodal values are used for the efforts that remains the normal bending moment m_n and the equivalent shear force V_n. Some examples are presented and the results compared with those of the finite element analysis.

1 INTRODUCTION

Many works about the use of the boundary element method for plate bending analysis have been published[1-8]. In all formulations presented in these papers, the integral representations of transversal displacement w and its derivative in the normal direction, $\partial w/\partial n$, for all nodes at the boundary of the plate are written. These formulations are adequate for the analysis of single plates or plates with rigid columns and beam connections, but for the analysis of plates coupled with flexible beams and columns they are not adequate, mainly in the connections along the boundary of the plate, because these structural elements need 3 nodal values to be represented, i.e. w, $\partial w/\partial n$ and $\partial w/\partial t$. Although in the usual boundary element formulations only w and $\partial w/\partial n$, appears as parameters, the inclusion of an extra nodal value in the derivative of the displacement can be easily implemented.

This work presents a boundary element formulation for plate bending analysis in which the boundary equation of displacements and its derivative in directions normal and tangential to the boundary are used. This formulation considers three nodal values in displacements: w, $\partial w/\partial n$ and $\partial w/\partial t$, for each boundary node but, according to Kirchhoff's hypothesis, only two nodal values are used for the efforts that remains the normal bending moment m_n and the equivalent shear force V_n. In the case of plates supported or clamped along their edges, this formulation collapses to the usual one. As this formulation considers three nodal values for displacements for each node at the boundary it is suitable for plate-beam-column coupling.

2 PLATE BENDING EQUATIONS

The integral equations for displacements at an internal point of a plate with domain "Ω" and boundary "Γ", on which a distributed load "g" is applied can be easily obtained from Betti's relation:

$$\int_V \sigma_{ij} \epsilon^*_{ij} dV = \int_V \sigma^*_{ij} \epsilon_{ij} dV \tag{1}$$

where V is the volume of the three-dimensional body and σ_{ij}, ϵ_{ij}, σ^*_{ij} and ϵ^*_{ij} are the stresses and strain for the real problems and the corresponding fundamental solutions. The integral in equation (1) becomes area integral by assuming small thickness and performing the integral along that dimension, as follows:

$$\int_\Omega m_{ij} \frac{\partial^2 w^*}{\partial x_i \partial x_j} d\Omega = \int_\Omega m^*_{ij} \frac{\partial^2 w}{\partial x_i \partial x_j} d\Omega \tag{2}$$

where m_{ij} and w are , respectively, the bending moment tensor and the transversal displacement of the plate. After integrating equation (2) by parts and applying the concept of Dirac's Delta to integrate over the domain the term in w results:

$$K(s)\, w(s) + \int_\Gamma \left[q^*_n(s,Q)\, w(Q) - m^*_n(s,Q) \frac{\partial w}{\partial n}(Q) - m^*_{ns}(s,Q) \frac{\partial w}{\partial t}(Q) \right] d\Gamma(Q) =$$

$$\int_\Gamma \left[q_n(Q)\, w^*(s,Q) - m_n(Q) \frac{\partial w^*}{\partial n}(s,Q) - m_{ns}(Q) \frac{\partial w^*(s,Q)}{\partial t} \right] d\Gamma(Q) +$$

$$+ \int_{\Omega_g} g(q)\, w^*(S,q)\, d\Omega_g(q) \tag{3}$$

The efforts q_n and m_{ns} in the second member of equation (3) can be expressed as a function of V_n, the Kirchhoff equivalent shear force, and the reactions R_{ci} at the corners of the plate as follows:

$$\int_\Gamma \left[q_n(Q)\, w*(s,Q) - m_{ns}(Q) \frac{\partial w^*}{\partial t}(s,Q) \right] d\Gamma = \int_\Gamma V_n(Q)\, w^*(s,Q)\, d\Gamma + \\ + \sum_{i=1}^{Nc} R_{ci}(q)\, w^*_{ci}(s,Q) \tag{4}$$

From (3) and (4) results:

$$K(s)\,w(s) + \int_\Gamma \left[q_n^*(s,Q)\, w(Q) - m_n^*(s,Q) \frac{\partial w}{\partial n}(Q) - m_{ns}^*(s,Q) \frac{\partial w(Q)}{\partial t} \right] d\Gamma(Q) =$$

$$\int_\Gamma \left[V_n(Q)\, w^*(s,Q) - m_n(Q) \frac{\partial w^*}{\partial n}(s,Q) \right] d\Gamma(Q) + \sum_{i=1}^{Nc} R_{ci}(Q)\, w^*_{ci}(s,Q) +$$

$$\int_{\Omega_g} g(q)\, w^*(S,q)\, d\Omega_g(q) \tag{5}$$

in which

K(s)=1 for a point "*s*" into plate domain
K(S)=$\beta/2\pi$ for a point "*S*" at a boundary corner, with internal angle β
K(S)=1/2 for a point "*S*" on a smooth boundary
$R_{ci}=m_{ns}^- - m_{ns}^+$ is the corner reaction

From equation (5) the integral representation of the derivative of the displacement in relation to a direction *m* of a systen of coordinates *(m,u)* can be derived as follows:

$$K_1(S) \frac{\partial w}{\partial m_s}(S) + K_2(S) \frac{\partial w}{\partial u_s}(S) + \int_\Gamma \left[\frac{\partial q_n^*}{\partial m_s}(S,Q)\, w(Q) - \frac{\partial m_n^*}{\partial m_s}(S,Q) \frac{\partial w}{\partial n}(Q) + \right.$$

$$\left. - \frac{\partial m_{ns}^*}{\partial m_s}(S,Q) \frac{\partial w}{\partial t}(Q) \right] d\Gamma(Q) = \int_\Gamma \left\{ V_n(Q) \frac{\partial w^*}{\partial m_s}(S,Q) - \right.$$

4 Boundary Element Technology

$$-m_n(Q)\frac{\partial}{\partial m_S}\left[\frac{\partial w^*}{\partial n}(S,Q)\right]\Bigg\}d\Gamma(Q)+\sum_{i=1}^{N_c}R_{ci}(Q)\frac{\partial w^*_{ci}}{\partial m_S}(S,Q)+$$

$$+\int_{\Omega_g}g(q)\frac{\partial w^*}{\partial m_S}(S,q)\,d\Omega_g(q) \qquad (6)$$

in which

$$K_1(S)=\frac{\beta}{2\pi}+\frac{1+\nu}{8\pi}[\sin(2\gamma)-\sin 2(\gamma+\beta)] \qquad (7)$$

$$K_2(S)=\frac{1+\nu}{8\pi}[\cos 2\gamma-\cos 2(\gamma+\beta) \qquad (8)$$

where γ is the angle between systems ***(n,t)*** and ***(m,u)***.

In order to solve numerically these integral equations, the boundary of the plate is divided into a series of segments, called boundary elements, in whose domain an approximation function is adopted for efforts and displacements. In the formulation used in this paper, displacements w, $\partial w/\partial n$, and $\partial w/\partial t$ and bending moment m_n are approximated by a linear function, and effective shear force V_n, on the boundary is approximated by concentrated reactions R_k applied to the element nodes [6].

By writing the equations of the displacement and its derivatives in normal and tangential directions for all nodes at the boundary and by performing all the integrations, the following set of linear equation can be obtained:

$$[H]\{w_\Gamma\}=[G]\{V_\Gamma\}+\{p\} \qquad (9)$$

in which $\{w_\Gamma\}$ and $\{V_\Gamma\}$ are vectors of nodal values of displacements and efforts at the boundary nodes.

The nodal value $\partial w/\partial t$ is null in all nodes along supported or clamped sides of a plate. In this case the only way of imposing these boundary conditions is by eliminating the equation related to these nodal values. So, if a plate is supported or clamped along the whole boundary, after imposing the boundary conditions, this formulation collapses to the usual one. After taking into account all the boundary conditions, equation (9) becomes:

$$[A]\{X\}=\{B\} \tag{10}$$

in which *{X}* is a vector defined by unknowns. After solving equation (10), displacements and curvatures at any point of the plate can be computed from equation (5), with $K(S)=1$.

3 Examples

The formulation presented in this paper was initially tested with plates totally supported or clamped along the boundary. In these examples the results of the analysis were the same as those from the usual formulation. In the analysis of plates with free edges with the proposed formulation, the results are very close to those obtained with finite element analysis using a very refined finite element mesh. The finite element used is the ***DKQ_4T[7]***, a quadrilateral finite element very effective for plate bending analysis. The results of the boundary element analysis are very close to those from finite element ones. The differences in these results for points in the domain of the plate are smaller than those along the boundary, but it should be noted that for the finite element the interpolation function for displacements along its sides is cubic and for the boundary element method a linear variation of the displacement is assumed.

Initially the results of the analysis of a square plate supported at its corners and with a uniform transversal load g, as shown in figure (1.a), are presented. For this analysis the Poisson's ratio is assumed to be $\nu=0.3$. For the boundary element analysis, the boundary was divided into 40 equal elements and for the finite element analysis a mesh of 400 square finite elements was used. Figures (2) and (3) show the vertical displacements w and $\partial w/\partial s$ along S_1 and the displacement w and the bending moment m_x along S_2.

The next example is a square plate clamped and supported in two adjacent sides and free in the other two, as shown in fig. (1.b). Poisson's ratio and meshes are the same used in the previous example. Figure (4) shows the vertical displacement w and the bending moment m_x along S_2. These results show that the proposed formulation is very effective for the analysis of bending plates with free edges.

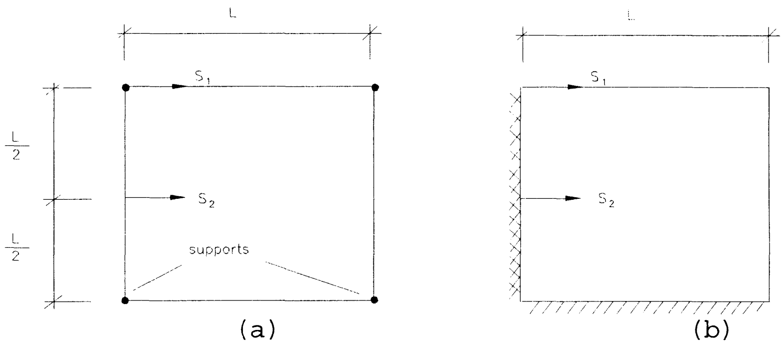

Fig. 1 - Plates analysed

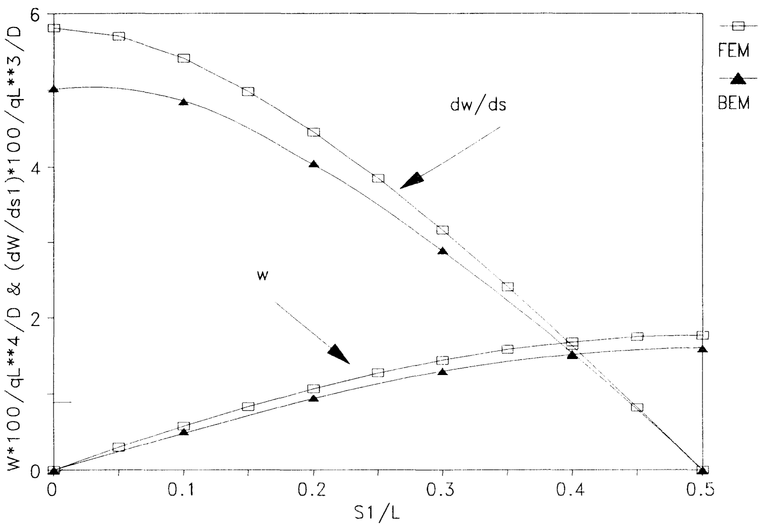

Fig. 2 - Transversal displacement and its derivative along S_1 for the plate supported at the corners

Transactions on Modelling and Simulation vol 9, © 1995 WIT Press, www.witpress.com, ISSN 1743-355X

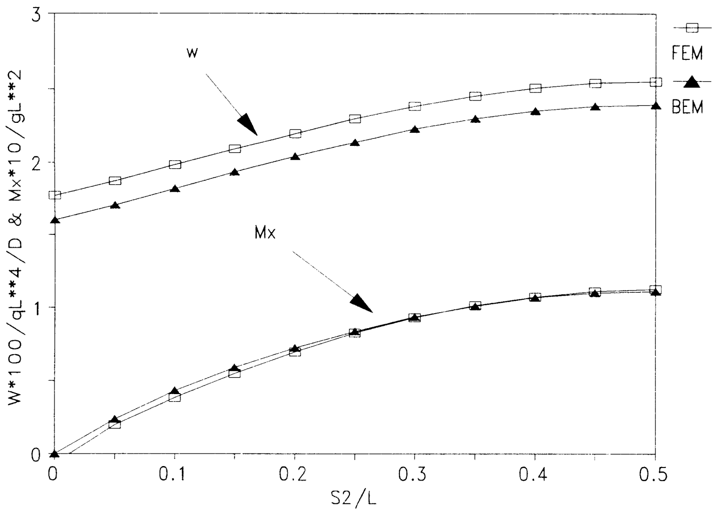

Fig. 3 - Transversal displacement and bending moment along S_2

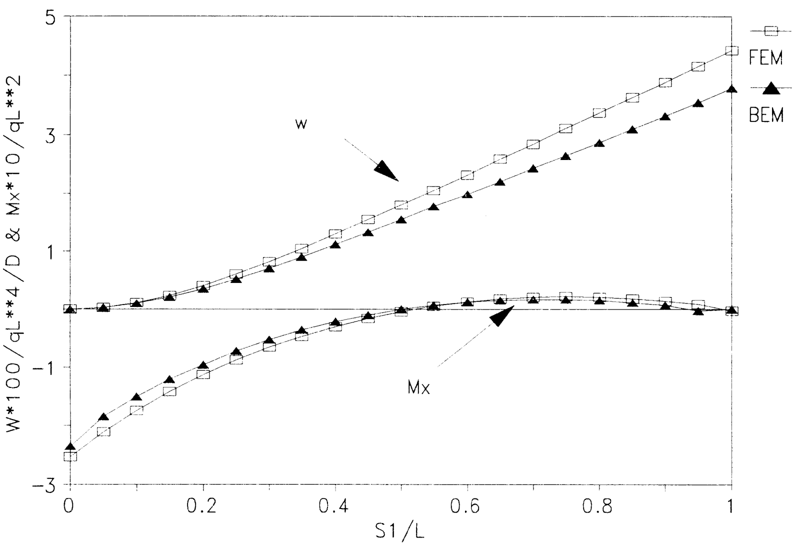

Fig. 4 - Transversal displacement and bending moment m_x along S_2

CONCLUSIONS

This work presents a boundary element formulation for plate bending analysis in which the boundary equations of the tranversal displacement and its derivatives in normal and tangential directions are used. This formulation allow the use of functions to represent $\partial w/\partial s$ along the boundary and collapses to the usual ones for plates totally supported or clamped along the boundary. It is very useful to analyze plates with free boundaries, as seen in the results presented in this paper.

REFERENCES

[1] - BEZINE, G.P. Boundary integral formulation for plate flexure with arbitrary boundary conditions. Mechanics Research Communications, 5 (4): 197-206, 1978

[2] - STERN, M., "A general boundary integral formulations for the numerical solution of plate bending problems", Int. J. Sol. Struct., Vol. 15, pp. 769-782, 1979.

[3] - KAMIYA, N.; SAWAKY, Y. An integral equation approach to finite deflections of elastic plates. Int. J. Non-Linear Mechanics, 17(3) : 187-194, 1982

[4] - KATSIKADELIS, J.T.; ARMENAKAS, A.E. Plates on elastic foundations by BIE method. Journal of Engineering Mechanics, ASCE, 110 (7): 1086-1104, July, 1984

[5] - BÉZINE, G. A boundary integral equation method for plate flexure with conditions inside the domain. Int. J. for Numerical Methods in Engineering, 17: 1647-1657, 1981

[6] - PAIVA, J.B. "Boundary element formulation for plate analysis with special distribution of reactions along the boundary". Advances in Engineering Software and Workstations 13: 162-168, July 1991

[7] - Rezende, M.N. and PAIVA J.B., "A quadrilateral discret Kirchhoff Finite Element for building slab analysis", II International Conference in Computational Structures Technology, Athens, Sept. 1995