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The method of fundamental solutions for Poisson's equation

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ABSTRACT

We show how to extend the method of fundamental solutions (MFS) to solve Poisson's equation in 2D without boundary or domain discretization. To do this an approximate particular solution is found by approximating the right hand side by thin plate splines. The particular solution is then subtracted from the complete solution and then Laplace's equation is solved by the usual MFS. Numerical results are obtained for a number of standard boundary value problems with 3-4 figure accuracy attainable by solving fewer than 20 linear equations.

1 INTRODUCTION

One of the major advantages of the BEM over the Finite Element and Finite Difference methods is that only boundary discretization is usually required rather than the domain discretization needed in those other methods. However, if the differential equation to be solved is inhomogeneous, the BEM becomes less attractive because integral reformulations generally involve a domain integral whose evaluation may consume the majority of the computation time [1]. Moreover, if the boundary of the domain is complicated, boundary discretization can become quite complex, particularly in three dimensions. Consequently, it is of some interest to develop numerical methods which require neither domain nor boundary discretization. In the spirit of the BEM, the method of Fundamental Solutions (MFS) has been used for homogeneous elliptic equations and because of its simplicity and rapid convergence, it is of interest to extend this latter technique to solve inhomogeneous partial differential equations. In some cases where a known particular solution is available, this has been done [3], but no method for general inhomogeneous problems seems to have been given. As a consequence, it is the purpose of this paper to propose an algorithm which extends the MFS to inhomogeneous equations in such a way that neither boundary nor domain discretization is needed.

To be more specific, we consider solving well-posed boundary value problems for Poisson's equation

$$\Delta u = f \tag{1}$$

on a bounded domain D in $\mathbf{R}^2$. To solve (1) numerically, we first reduce it to Laplace's equation by subtracting a particular solution v from u. If $\hat{u} = u - v$, then $\hat{u}$ satisfies

$$\Delta \hat{u} = 0, \tag{2}$$

with boundary conditions for $\hat{u}$ derived from those for v. Once v is determined, $\hat{u}$ is obtained using the MFS. If f is simple, v may be calculated analytically. However, for general $f's$ only approximations for v are generally available [5]. In the past decade this approximation problem has been extensively studied in relation to the BEM where a number of alternatives to cell integration have been proposed. At present the most widely used technique seems to be the Dual Reciprocity Method (DRM) where f is approximated by a finite series of radial basis functions ($rbfs$) with an approximation $\hat{v}$ to v calculated analytically from $\hat{f}$. In much of the work on the DRM, the choice of $rbfs$ appears to be arbitrary. To improve the efficiency and reliability of rbf approximation, we have used a theorem of Duchon which suggests the use of thin plate splines as optimal basis functions [6, 7]. Our numerical results for two-dimensional problems bear this out.

The paper is divided into seven sections. In Section 2, we briefly review the MFS for solving Laplace's equation. In Section 3, we develop a method for approximating particular solutions to (1), with emphasis on the theory and use of rbf approximations. In Section 4 we outline our algorithms for solving (1) for two dimensional problems. In Section 5, we give some a-priori error estimates for our method for the Dirichlet problem showing that the L_2 error in approximating u can be bounded in terms of approximation errors of the data. To the best of our knowledge, these error estimates are new.

In Section 6 we give numerical results for a set of standard 2D problems. Comparison with analogous results for the BEM using standard DRM approximations and recent results of Allesandri and Tralli [8] using bicubic splines, shows that our approach is more accurate and more efficient to implement. We conclude with some directions for future work.

2 THE MFS FOR LAPLACE'S EQUATION

As we indicated in the Introduction, we will solve Poisson's equation by reducing it to Laplace's equation (2). To solve (2) we use the MFS, which has been shown to be a highly accurate and efficient numerical technique

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[2, 4]. For completeness, we summarize the essential aspects of this method here.

Let D be a bounded simply connected set in $\mathbf{R}^2$ with boundary S (this restriction can be relaxed) and let S_i, $i = 1, 2, \cdots, m$ be a partition of S. That is $S = \cup_{i=1}^{m} S_i$ and $S_i \cap S_j = \emptyset$, $i \neq j$. We consider solving the boundary value problem

$$\Delta u(P) = 0, \qquad P \in D, \tag{3}$$

with boundary conditions

$$B_i(u)\,|_{S_i} = 0, \quad i = 1, 2, \cdots, m, \tag{4}$$

where $\{B_i\}_{i=1}^{m}$ are differential operators (possibly nonlinear) and $B_i(u)\,|_{S_i}$ is the restriction of $B_i(u)$ to S_i. We assume that (3)-(4) has a unique solution.

To approximate the solution to (3)-(4) we use the MFS. For this, let

$$G(P, Q) = \log \|P - Q\| / 2\pi, \tag{5}$$

$\left(P = (x_1, y_1), Q = (x_2, y_2), \|P - Q\| = \left[(x_1 - x_2)^2 + (y_1 - y_2)^2\right]^{1/2}\right)$ be the fundamental solution to $\Delta u = 0$, and let

$$H(D) = \{u : \Delta u(P) = 0,\ P \in D\} \tag{6}$$

be the set of harmonic functions in D. Let T be a curve in $\mathbf{R}^2$ with D in its interior. Assume that $\{Q_k\}_{k=1}^{\infty}$ is a dense set of points in T and let

$$H'(D) = \operatorname{span}\{G(P, Q_k)\}_{k=1}^{\infty} \cup \mathbf{R},\ P \in D. \tag{7}$$

If D is connected and S is a Lyapounov curve, then it was shown in [2] that $H'(D)$ is dense in $H(D)$. Thus every harmonic function u can be approximated arbitrarily closely by a function of the form

$$u_n(P) = \sum_{k=1}^{n} c_k G(P, Q_k) + c. \tag{8}$$

In [3] Bogomolny improved on the results in [6] and showed that T could be taken as a circle with radius R and that $\{c_k\}_{k=1}^{n} \cup \{c\}$ could be chosen so that u_n converges exponentially fast in n and R if u and S are analytic.

The above results suggest that an efficient algorithm for solving (3) can be obtained by looking for approximations to u in the form (8) and then trying to satisfy the boundary conditions in some fashion. The simplest way of doing this is to choose n_i points $\{P_j\}$ in S_i with $\sum_{i=1}^{m} n_i = n + 1$ and then satisfying the boundary conditions by collocation; i.e. by choosing $\{c_k\}_{k=1}^{n}$ and c to satisfy

$$B_\ell(u_n(P_j)) = 0,\ j = 1, 2, \cdots, n_\ell,\ \ell = 1, 2, \cdots, m. \tag{9}$$

For instance, if we have Dirichlet boundary conditions, then $B_\ell(u) = u - g$, $\ell = 1, 2, \cdots, m$ where g is a given continuous function on S, and (9) becomes

$$\sum_{k=1}^{n} c_k G(P_j, Q_k) + c = g(P_j), \; j = 1, 2, \cdots, n+1. \tag{10}$$

Equations (9) are then solved for $\{c_k\}_{k=1}^{m} \cup \{c\}$ and u is approximated by (8).

In contrast to previous work, we have found the MFS to be quite sensitive to the inclusion of the constant term in u_n. Although Bogomolny's theoretical results suggest that a constant be included, his numerical results indicated little effect. On the other hand, we have found that the value of c to be quite dependent on the eccentricity of the domain. In our experiments with elliptical domains, we have found that $|c|$ generally increases as the eccentricity increases.

3 COMPUTATION OF PARTICULAR SOLUTIONS

To compute particular solutions to Poisson's equation we consider the approach used in the DRM, that of approximating f in (1) as a linear combination of globally defined basis functions $\{f_k\}_{k=1}^{n}$

$$f \simeq \sum_{k=1}^{n} a_k f_k \equiv \hat{f}, \tag{11}$$

and then obtaining an approximate particular solution $\hat{v}$ by

$$\hat{v} = \sum_{k=1}^{n} a_k \hat{v}_k \tag{12}$$

where $\hat{v}_k$ satisfies

$$\Delta \hat{v}_k = f_k. \tag{13}$$

If $\{\hat{v}_k\}_{k=1}^{n}$ can be obtained analytically, then only the expansion coefficients $\{a_k\}_{k=1}^{n}$ need be determined. The simplest way of doing this seems to be via interpolation: that is a set of n points $\{P_j\}_{j=1}^{n}$ is chosen in the domain of f and then setting

$$\hat{f}(P_j) = f(P_j), \quad i = 1, 2, \cdots, n. \tag{14}$$

Doing this gives the n equations

$$\sum_{k=1}^{n} a_k f_k(P_j) = f(P_j), \quad i = 1, 2, \cdots, n, \tag{15}$$

for $\{a_k\}_{k=1}^{n}$. If the matrix $F = [f_k(P_j)]_{(j,k)=1}^{n}$ is non-singular, then (15) can be solved uniquely for $\{a_k\}_{k=1}^{n}$.

The question now arises as to the choice of basis functions $\{f_k\}_{k=1}^n$ and interpolation points $\{P_j\}_{j=1}^n$. If we argue in analogy to the one-dimensional case, then trigonometric, polynomial and piecewise polynomial functions immediately come to mind, and all of these have been discussed in the literature [5, 8]. However, as is well-known, in one dimension naive choices of basis functions and interpolation points, such as interpolation using monomials with equally spaced points, can be disastrous [9].

In one dimension cubic splines provide a solution to some of the problems of polynomial interpolation because they give the smoothest interpolants in the sense of minimizing the "curvature" integral [9]

$$\int_{-\infty}^{\infty} |f''(x)|^2 \, dx. \tag{16}$$

From this one might expect that tensor products of cubic splines would be optimal interpolants in $\mathbf{R}^2$. Surprisingly, this is not the case [6, 7]. By a theorem of Duchon [6], optimal interpolants in $\mathbf{R}^2$ are given by the thin plate splines [6, 7]. Specifically we have the following theorem.

Theorem 1. (Duchon) Let $\tau(P) \in C^2(\mathbf{R}^2 - \{0\})$, be such that

$$I(\tau) = \int_{R^2} \sum_{j=1}^{2} \sum_{i=1}^{2} \left[\frac{\partial^2 \tau(P)}{\partial x_i \partial x_j} \right]^2 d\mathbf{x} < \infty, \tag{17}$$

and let $f : \mathbf{R}^2 \to \mathbf{R}$ have values $f(P_i)$, $i = 1, 2, \cdots, n$ $(n \geq d+1)$ on the noncollinear points $\{P_i\}_{i=1}^n$. Then there exists a unique τ interpolating to f at $\{P_i\}_{i=1}^n$ and minimizing $I(\tau)$ iff $(P = (x, y))$

$$\tau(P) = \sum_{j=1}^{n} \lambda_j \|P - P_j\|^2 \log \|P - P_j\| + a + bx + cy, \tag{18}$$

where

$$\sum_{j=1}^{n} \lambda_j = \sum_{j=1}^{n} \lambda_j x_j = \sum_{j=1}^{n} \lambda_j y_j = 0. \tag{19}$$

■

The functions $\tau(P)$ in Theorem 1 are called *thin plate splines* and are the multidimensional analogues of cubic splines. Their optimal interpolating properties suggest taking

$$f_k = \|P - P_k\|^2 \log \|P - P_k\|, \quad k = 1, 2, \cdots, n-2, \; f_{n-2} = 1, \; f_{n-1} = x, \; f_n = y. \tag{20}$$

We note in (20) that the basis functions are similar to the basis functions $f_k = \|P - P_k\| + 1$ commonly used in the DRM. Both of these sets of functions are particular cases of radial basis functions [7]; i.e. functions of the form

$$f(P) = \varphi(\|P\|) + p_m(P) \tag{21}$$

where φ: $\{x \geq 0\} \to \mathbf{R}$ is continuous with $\varphi(0) \geq 0$ and p_m is a polynomial of degree m. In contrast to some statements in the literature [8], there exists an extensive approximation theory for these functions [7] and Duchon's result supports the choice of $rbfs$ as interpolants in the DRM. Moreover, there exist estimates for the interplation error for the thin plate splines which leads us to advocate their use in the BEM and related algorithms. For instance it is shown in [7] that $0(h^4)$ error can be obtained by interpolating f by thin plate splines on a uniform mesh.

If the thin plate splines are used to approximate f, then we can obtain particular solutions $\hat{v}_k$ in the following way: we integrate Poisson's equation in polar co-ordinates to get $(k = 1, 2, \cdots, n-2)$

$$\hat{v}_k(P) = \Psi\left(\|P - P_k\|\right), \tag{22}$$

where Ψ satisfies

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Psi}{\partial r}\right) = r^2 \log r. \tag{23}$$

Successive integration of (23) gives

$$\Psi = \frac{r^4 \log r}{16} - \frac{r^3}{32}. \tag{24}$$

For $k = n-2, n-1, n$, we have

$$\hat{v}_{n-2} = \frac{x^2 + y^2}{4}, \hat{v}_{n-1} = \frac{x^3}{6}, \hat{v}_n \equiv \frac{y^3}{6}. \tag{25}$$

4 THE MFS FOR POISSON'S EQUATION

We now combine the results of Section 2 and 3 to give MFS algorithms for solving Poisson's equation. If an exact particular solution is not available, we begin by finding an approximate particular solution $\hat{v}$ by using the algorithm discussed previously. Among other things, this enables us to compare our results with existing DRMBEM algorithms with $\varphi(r) = 1 + r$.

Having obtained $\hat{v}$ define $\hat{w}$ as the solution to

$$\Delta \hat{w} = 0, \quad B_i\left(\hat{w} + \hat{v}\right)\big|_{S_i} = 0, \quad i = 1, 2, \cdots, m. \tag{26}$$

To solve (26) we use the MFS with n sources uniformly distributed around a circle of radius R surrounding D. If S has a smooth parameterization $S = \{(a(\theta), b(\theta)), 0 \leq \theta < L\}$, then we collocate at the points

$$\left\{\left[a\left(\frac{jL}{n}\right), b\left(\frac{jL}{n}\right)\right] : j = 0, 1, 2, \cdots, n-1\right\}. \tag{27}$$

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At present these are the only examples we have examined. The resulting approximation to (26) is denoted by $\bar{w}$ and then u is approximated by

$$\hat{u} = \bar{w} + \hat{v}. \tag{28}$$

Since $\Delta\bar{w} = 0$ and $\Delta\hat{v} = \hat{f}$ by construction,

$$\Delta\hat{u} = \Delta\bar{w} + \hat{v} = \hat{f}, \tag{29}$$

a result needed in the following section.

5 SOME ERROR ANALYSIS FOR THE MFS

With the exception of Cheng's MFS convergence result for the Dirichlet problem in $\mathbf{R}^2$, no general convergence analysis seems to have been given for the MFS [4]. However, using some standard a-priori estimates for solutions to Poisson's equation it is possible to obtain some heuristic error bounds for $\hat{u}$ in terms of the data. For simplicity we confine our argument to the Dirichlet problem.

For our analysis we assume that $u \in C^2(D) \cap C^0(D \cup S)$ and that S, f and g are smooth enough so that there exists a unique solution to

$$\Delta u = f, \quad u|_S = g. \tag{30}$$

For $w \in C^2(D) \cap C^0(D \cup S)$ we have the well-known inequality [10]

$$\int_D w^2(Q)dQ \le c_1 \int_S w^2(Q)dQ + c_2 \int_D [\Delta w(Q)]^2\, dQ \tag{31}$$

where c_1 and c_2 are constants not depending on w. If we let $w = u - \hat{u}$ in (31) then

$$\begin{aligned} \textstyle\int_D (u-\hat{u})^2\, dQ &\le c_1 \textstyle\int_S (u-\hat{u})^2\, dQ + c_2 \textstyle\int_D (\Delta u - \Delta\hat{u})^2\, dQ \\ &= c_1 \textstyle\int_S (u-\hat{u})^2\, dQ + c_2 \textstyle\int_D \left(f-\hat{f}\right)^2 dQ \end{aligned} \tag{32}$$

since $\Delta u = f$ and $\Delta\hat{u} = \hat{f}$ by (29).

Now on S, $u|_S = g$ and $\hat{u}|_S = \bar{w}\,|_S + \hat{v}\,|_S$. By construction $\bar{w}$ is chosen to interpolate $g - \hat{v}$ on S by a linear combination of potentials. Thus $\bar{w}\,|_S = g_I - \hat{v}_I$ and $(u - \hat{v})|_S = g - g_I + \hat{v}_I - \hat{v}|_S = g - g_I - (\hat{v}\,|_S - \hat{v}_I\,|_S)$. Using this in the first integral in (32) gives

$$\int_D (u-\hat{u})^2\, dQ \le c_1 \int_S [(g-g_I) - (\hat{v} - v_I)]^2\, dQ + c_2 \int_D \left(f - \hat{f}\right)^2 dQ. \tag{33}$$

Using the triangle inequality in the first integral and taking square roots gives the L_2 error bound

$$\|u - \hat{u}\|_D \le c_1' \left[\|g - g_I\|_S + \|\hat{v} - \hat{v}_I\|_S\right] + c\left\|f - \hat{f}\right\|_D \tag{34}$$

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where

$$\|w\|_D = \left(\int_D w^2 dQ\right)^{1/2} \text{ and } \|w\|_S = \left[\int_S w^2 dQ\right]^{1/2}. \tag{35}$$

As can be seen, (34) gives an error bound for $\hat{u}$ in terms of approximation errors to the data (f, g) in (30). If the data are analytic, then (34) and our arguments in Section 2 and 3 suggest that

$$\|u - \hat{u}\|_D = \|e\|_D = E(n, R) + 0\left(h^4\right) \tag{36}$$

where $E(n, R) \to 0$ exponentially fast in (n, R). In this case, we expect the error in $\hat{u}$ to be dominated by the approximation error in $\hat{f}$.

6 NUMERICAL EXAMPLES

To validate the theory developed in Sections 2-5 we solved Poisson's equation on the ellipse $x^2/4 + y^2 \leq 1$ with $f = -2, -x, -x^2, 4 - x^2$ and $u|s = 0$. These problems were chosen so we could compare our results with previously published BEM results in [5, 8]. Because of lack of space we present results only for $f = -x^2$ in Table 1. As can be seen, they are generally almost an order of magnitude more accurate than those in [5, 8] and required fewer arithmetic operations because there were no numerical integrations to perform.

The calculations were done using 16 sources evenly spaced around a circle of radius eight to solve equation (2). The approximate particular solution was obtained using a thin plate spline interpolant on the 33 points given in [5]. All numbers have been rounded to 3 decimal places.

In Table 1 the analytical solution is given by $[(-50x^2 - 8y^2 + 33.6)(x^2/4 + y^2 - 1)/246]$ [5]. The DRM results are those in [5] using 17 linear boundary elements and $\{1+\|P - P_k\|\}$ as the basis functions to approximate $-x^2$. The values in the last column are those obtained in [8] using the BEM with 17 linear elements and bicubic Hermite interpolation to approximate $-x^2$.

Table 1. Solutions of $\triangle u = -x^2$ on $x^2/4 + y^2 \leq 1$

x	y	Analytical Solution	MFS Solution	DRM Solution	H-bicubic Solution
1.5	.00	.260	.261	.269	.259
1.2	-.35	.220	.220	.220	.224
0.6	-.45	.144	.144	.135	.140
0.0	-.45	.103	.104	.092	.097
0.9	0.00	.240	.240	.236	.235
0.3	0.00	.151	.151	.142	.149
0.0	0.00	.137	.135	.127	.132

7 CONCLUSIONS

We have shown how to extend the MFS to solve Poisson's equation without boundary or domain discretization. Numerical results are generally superior to those obtained by related BEMDRM methods. Although we have only considered solving Poisson's equation in this paper, the method can be applied to more general linear and nonlinear partial differential equations. Work is currently in progress on these generalizations.

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