The Complex Variable Boundary Element Method for potential flow problems

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Abstract

The Cauchy type integral, used to represent the complex velocity, is converted to a line distribution of sources and vortices in the complex plane. The specification of the normal velocity on the bounding contour leads to a Riemann-Hilbert problem, which provides the theoretical foundation of the method. The boundary element discretization results in a simple algorithm for calculating potential flows in multiple-connected domains. Flow problems with periodic or homogeneous outer boundary conditions are treated using the concept of the Green's function in the complex plane.

1 Introduction

The Complex Variable Boundary Element Method (CVBEM) evolved as a numerical procedure for solving boundary value problems for analytic functions in terms of discretized Cauchy type integrals [1]. The method described in this paper is based on the same principle. However, instead of linking the integral to the complex potential, as is commonly done, it is linked to the complex velocity. The main advantages of this approach are: 1) the complex velocity is single-valued and hence no cuts in the computational domain are necessary and 2) the representing Cauchy type integral is equivalent to the contour distribution of source and vortex singularities. By selecting the Cauchy density as the boundary value of a function analytic in the external flow region, it is possible to specify the far field condition such that there is no flow in the complementary interior domain. In this particular case the normal and tangential velocities become decoupled, corresponding to the source and vortex densities respectively. The imposition of the normal-velocity boundary condition and the subsequent discretization by boundary elements leads to the vortex panel method in the complex plane, reported earlier [2].

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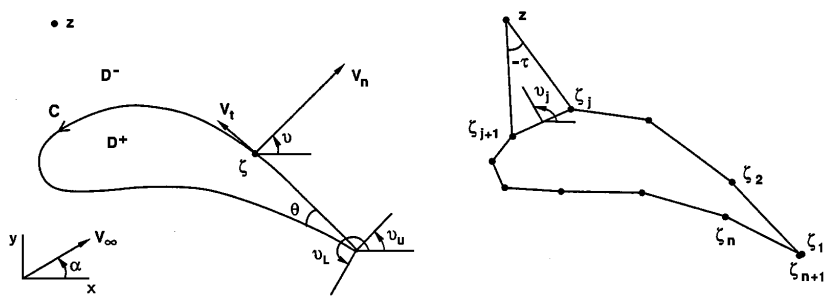
2 Cauchy integral formulation

The complex disturbance velocity is represented by the Cauchy type integral

$$w(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta, \tag{1}$$

where z and ζ are the complex coordinates of the observation and contour points respectively and C is the counterclockwise oriented, simple closed airfoil contour, Fig. 1a. The Cauchy density f is a continuous, complex-valued function defined on C. For multi-component airfoils, C is a union of nonintersecting simple-closed contours.



(a) geometrical description (b) approximation polygon

Figure 1: Single airfoil.

Introducing the angle ν between the outward normal to C and the real axis, the integral of Eq. (1) can be converted to the contour distribution of sources and vortices

$$w(z) = \int_C \left[\frac{\sigma(\zeta)}{2\pi(z-\zeta)} + \frac{i\gamma(\zeta)}{2\pi(z-\zeta)} \right] |d\zeta|, \tag{2}$$

using

$$d\zeta = [-\sin\nu(\zeta) + i\cos\nu(\zeta)]|d\zeta| = ie^{i\nu(\zeta)}|d\zeta| \tag{3}$$

and

$$f(\zeta) = -[\sigma(\zeta) + i\gamma(\zeta)]e^{-i\nu(\zeta)}. \tag{4}$$

The real-valued functions σ and γ are the source and vortex densities respectively.

There is a multiplicity of density functions f capable of representing a given analytic function w in either the internal domain D^+ (to the left of C) or the external domain D^- (to the right of C). For an external flow problem, it is natural to specify $f(\zeta)$ as the boundary value of a function $f(z)$ analytic in D^- and continuous in $D^- \cup C$. Applying the Cauchy integral formula to Eq. (1), we obtain

$$w(z) = \begin{cases} f(\infty) - f(z), & z \in D^- \cup C, \\ f(\infty), & z \in D^+, \end{cases} \tag{5}$$

where $f(\infty)$ is the value of $f(z)$ as $|z| \to \infty$.

If the free stream velocity is of unit magnitude and angle α to the real axis, the (total) complex velocity will be

$$W(z) = e^{-i\alpha} + w(z). \tag{6}$$

Choosing $f(\infty) = -e^{-i\alpha}$, it follows from Eqs. (5) and (6) that the fictitious interior flow vanishes whereas on the exterior face of the contour

$$W(\zeta) = -f(\zeta). \tag{7}$$

From Fig. 1a it can also be verified that in terms of the normal and tangential components

$$W(\zeta) = [V_n(\zeta) - iV_t(\zeta)]e^{-i\nu(\zeta)}. \tag{8}$$

Substituting Eq. (4) in (7) and comparing the latter with Eq. (8) shows that

$$V_n(\zeta) = \sigma(\zeta) \quad \text{and} \quad V_t(\zeta) = -\gamma(\zeta). \tag{9}$$

Accordingly, the discharge and circulation constants are given by

$$Q = \int_C V_n|d\zeta| = \int_C \sigma(\zeta)|d\zeta| \quad \text{and} \quad \Gamma = \int_C V_t|d\zeta| = -\int_C \gamma(\zeta)|d\zeta| \tag{10}$$

3 Airfoil boundary value problem

We assume that the normal component of velocity $V_n \equiv \sigma$ is prescribed (typically as zero) on the closed airfoil contour C. From Eqs. (6) and (8) the following boundary condition is obtained for the complex disturbance velocity:

$$\text{Re}\big\{[e^{-i\alpha} + w(\zeta)]e^{i\nu(\zeta)}\big\} = \sigma(\zeta). \tag{11}$$

Equation (11), written in the form

$$\text{Re}\left\{\frac{w(\zeta)}{q(\zeta)}\right\} = c(\zeta), \tag{12}$$

where

$$q(\zeta) = e^{-i\nu(\zeta)} \quad \text{and} \quad c(\zeta) = \sigma(\zeta) - \cos[\nu(\zeta) - \alpha], \tag{13}$$

are functions prescribed on the contour C, specifies the Riemann-Hilbert problem for the analytic function w. It resembles the Schwarz problem, except that the term in the curly brackets of Eq. (12) is not a boundary value of a function analytic in

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D^-. Using Gakhov's regularization method [3], it can be shown that the solution can be made unique by specifying the circulation Γ if C is smooth [4] or by requiring that w be bounded if C possesses a single corner point [5], also known as the trailing edge. Since a discontinuity of the density function f gives rise to a logarithmic singularity in the integral of Eq. (1), this (Kutta-Joukowski) condition is synonymous with the requirement that f be equal on the upper and lower sides of the trailing edge: $f_U = f_L$. For the trailing edge angle

$$\theta = \nu_L - \nu_U - \pi, \qquad \pi > \theta > 0 \tag{14}$$

we obtain from Eq. (4) the conditions

$$\gamma_U = \frac{\sigma_L + \sigma_U \cos\theta}{\sin\theta} \quad \text{and} \quad \gamma_L = -\frac{\sigma_U + \sigma_L \cos\theta}{\sin\theta}. \tag{15}$$

Although an explicit solution of Eqs. (12)–(13) exists [4], its evaluation involves conformal mapping as an intermediary step and hence is less practical than a direct numerical solution obtained by the CVBEM.

4 The CVBEM algorithm

The airfoil contour is approximated by n straight-line boundary elements, as indicated in Fig. 1b. The nodal points, which are the vertices of the approximation polygon, are numbered counterclockwise, $1, 2, \ldots, n+1$, starting with the upper trailing edge point, ζ_1 and ending with the lower trailing edge point, ζ_{n+1}. The jth boundary element is the line segment between points ζ_j and ζ_{j+1}. For a closed trailing edge, $\zeta_{n+1} = \zeta_1$.

The Cauchy type integral of Eq. (1) is discretized as

$$w(z) = \sum_{j=1}^{n} \Delta_j w(z), \tag{16}$$

where

$$\Delta_j w(z) = \frac{1}{2\pi i} \int_{\zeta_j}^{\zeta_{j+1}} \frac{f(\zeta)}{\zeta - z} d\zeta \tag{17}$$

is the contribution of the jth boundary element to the complex disturbance velocity at the observation point z.

In the present method, the density function f between the boundary element end points ζ_j and ζ_{j+1} is represented by the linear trial function

$$\begin{aligned} f(\zeta) &= f_j + \frac{f_{j+1} - f_j}{\zeta_{j+1} - \zeta_j}(\zeta - \zeta_j) \\ &= \frac{f_{j+1} - f_j}{\zeta_{j+1} - \zeta_j}(\zeta - z) + f_{j+1}\frac{z - \zeta_j}{\zeta_{j+1} - \zeta_j} - f_j \frac{z - \zeta_{j+1}}{\zeta_{j+1} - \zeta_j}, \end{aligned} \tag{18}$$

where f_j and f_{j+1} are the values of $f(z)$ at the respective end points of the boundary element. Substituting Eq. (18) in Eq. (17), we find

$$\Delta_j w(z) = \frac{f_{j+1} - f_j}{2\pi i} + \frac{1}{2\pi i}\left[f_{j+1}\frac{z - \zeta_j}{\zeta_{j+1} - \zeta_j} - f_j \frac{z - \zeta_{j+1}}{\zeta_{j+1} - \zeta_j} \right] \ln \frac{\zeta_{j+1} - z}{\zeta_j - z}. \tag{19}$$

From Eq. (4) we have for the jth boundary element

$$\begin{aligned} f_j &= -(\sigma_j + i\gamma_j)e^{-i\nu_j} \\ f_{j+1} &= -(\sigma_{j+1} + i\gamma_{j+1})e^{-i\nu_j}, \end{aligned} \tag{20}$$

where

$$e^{i\nu_j} = \cos\nu_j + i\sin\nu_j = -i\frac{\zeta_{j+1} - \zeta_j}{|\zeta_{j+1} - \zeta_j|}. \tag{21}$$

The logarithmic term of Eq. (19) can be written

$$\ln\frac{\zeta_{j+1} - z}{\zeta_j - z} = \ln\left|\frac{\zeta_{j+1} - z}{\zeta_j - z}\right| + i\tau,$$

where $-\pi \leq \tau \leq \pi$ is the angle obtained by the rotation of the vector $\zeta_j - z$ into the direction of the vector $\zeta_{j+1} - z$, see Fig. 1b. In the limit as the point z approaches an interior point of the jth boundary element from the flow field side D^-, τ tends to the value $-\pi$. Accordingly, the complex disturbance velocity induced by the jth segment at its own midpoint

$$z_j = \frac{1}{2}(\zeta_j + \zeta_{j+1}) \tag{22}$$

is

$$\Delta_j w(z_j) = \frac{f_{j+1} - f_j}{2\pi i} - \frac{1}{4}(f_{j+1} + f_j). \tag{23}$$

From Eqs. (16),(19),(20) and (23), we obtain for the complex disturbance velocity at the midpoint z_k of the kth segment

$$w(z_k) = \sum_{j=1}^{n+1} C_{k,j}(\sigma_j + i\gamma_j), \quad k = 1, \ldots, n. \tag{24}$$

The complex matrix $C_{k,j}$ can be evaluated as

$$C_{k,j} = K_{k,j} + L_{k,j} \tag{25}$$

where

$$K_{k,j} = \begin{cases} \frac{e^{-i\nu_j}}{2\pi i}\left(1 + \frac{z_k - \zeta_{j+1}}{\zeta_{j+1} - \zeta_j}\ln\frac{\zeta_{j+1} - z_k}{\zeta_j - z_k}\right), & j \neq k,\, n+1 \\ \left(\frac{1}{4} + \frac{1}{2\pi i}\right)e^{-i\nu_k}, & j = k \\ 0, & j = n+1 \end{cases} \tag{26}$$

$$L_{k,j} = \begin{cases} \frac{-e^{-i\nu_{j-1}}}{2\pi i}\left(1 + \frac{z_k - \zeta_{j-1}}{\zeta_j - \zeta_{j-1}}\ln\frac{\zeta_j - z_k}{\zeta_{j-1} - z_k}\right), & j \neq 1,\, k+1 \\ \left(\frac{1}{4} - \frac{1}{2\pi i}\right)e^{-i\nu_k}, & j = k+1 \\ 0, & j = 1 \end{cases} \tag{27}$$

Equation (11) is to be satisfied at all boundary element midpoints:

$$\mathrm{Re}\{[e^{-i\alpha} + w(z_k)]e^{i\nu_k}\} = \frac{1}{2}(\sigma_k + \sigma_{k+1}), \quad k = 1, \dots, n, \tag{28}$$

where σ_k and σ_{k+1} stand for the prescribed values of normal velocity at the end-points of the kth boundary element. Substituting from Eq. (24) and separating the terms containing the unknown values of vortex density from the given quantities, we obtain

$$\sum_{j=1}^{n+1} \mathrm{Im}\{e^{i\nu_k} C_{k,j}\}\gamma_j = \mathrm{Re}\left\{e^{i\nu_k}(e^{-i\alpha} + \sum_{j=1}^{n+1} C_{k,j}\sigma_j)\right\} - \frac{1}{2}(\sigma_k + \sigma_{k+1})$$
$$k = 1, \dots, n. \tag{29}$$

This represents a system of n linear algebraic equations in $n + 1$ unknown vortex densities γ_j. However, with the inclusion of the two trailing edge conditions of Eq. (23), we end up with $n + 2$ linear equations in the $n + 1$ unknown vortex densities.

Thus, adopting for Eq. (29) the matrix notation

$$A_{k,j}\gamma_j = b_k$$

and, inserting the trailing-edge conditions of Eq. (15) as the first and the last rows, we obtain

$$A_{1,j} = \begin{cases} 1, & j = 1 \\ 0, & j = 2, \dots, n+1, \end{cases} \quad b_1 = \frac{\sigma_{n+1} + \sigma_1 \cos\theta}{\sin\theta}$$

$$A_{k+1,j} = \mathrm{Im}\{e^{i\nu_k} C_{k,j}\}, \quad k = 1, \dots, n, \quad j = 1, \dots, n+1$$

$$b_{k+1} = \mathrm{Re}\left\{e^{i\nu_k}(e^{-i\alpha} + \sum_{j=1}^{n+1} C_{k,j}\sigma_j)\right\} - \frac{1}{2}(\sigma_k + \sigma_{k+1}), \quad k = 1, \dots, n$$

$$A_{n+2,j} = \begin{cases} 0, & j = 1, \dots, n \\ 1, & j = n+1, \end{cases} \quad b_{n+2} = -\frac{\sigma_1 + \sigma_{n+1} \cos\theta}{\sin\theta}.$$

The cosine and sine of the trailing edge angle θ are obtained according to Eq. (14) as the real and imaginary parts of

$$e^{i\theta} = e^{i(\nu_n - \nu_1 - \pi)} = \frac{\zeta_n - \zeta_{n+1}}{|\zeta_n - \zeta_{n+1}|} \frac{|\zeta_2 - \zeta_1|}{\zeta_2 - \zeta_1}.$$

An extension of the algorithm to a multi-component airfoil is fairly straightforward [2]. Denoting by $m(1)$, $m(2)$, …, the number of boundary elements on the airfoil components 1, 2, …, the numbering is such that ζ_1, ζ_2, …, $\zeta_{m(1)+1}$ are the corner points of the first component, $\zeta_{m(1)+2}$, $\zeta_{m(1)+3}$, …, $\zeta_{m(1)+m(2)+2}$ the

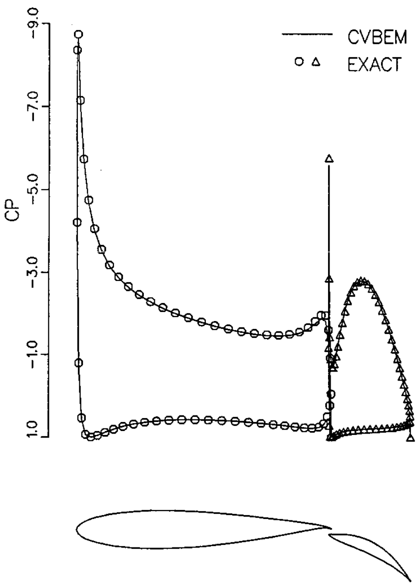


Figure 2: Williams airfoil.

corner points of the second component, and so on. The two trailing edge conditions (15) are satisfied on each airfoil component, leaving the system of linear equations overdetermined by the number of airfoil components. Several methods can be used to solve numerically such a system [2].

Once the values γ_j have been computed, the normal and tangential components of velocity at the corner points are obtained as equal to σ_j and $-\gamma_j$ respectively, see Eqs. (9). The corresponding pressure coefficients are obtained as

$$C_p(\zeta_j) = 1 - (\sigma_j^2 + \gamma_j^2). \tag{30}$$

A numerical example is given for Williams' two-component airfoil [6], configuration 'A', designed by conformal mapping. The number of boundary elements is $m(1) = m(2) = 62$, with the contour point coordinates listed in [6]. From Fig. 2 it is seen that the CVBEM reproduces the pressure distributions correctly, including the steep suction peaks.

5 Modified distributions of sources and vortices

If the external flow around a body is subject to periodic or outer constraints, which can be described by linear homogeneous boundary conditions, it is convenient to generalize the line distribution of sources and vortices as

$$w(z) = \int_C [\sigma(\zeta)G_\sigma(z,\zeta) + \gamma(\zeta)G_\gamma(z,\zeta)]|d\zeta|, \tag{31}$$

where C is the body contour and

$$G_\sigma(z,\zeta) = \frac{1}{2\pi(z-\zeta)} + H_\sigma(z,\zeta), \quad G_\gamma(z,\zeta) = \frac{i}{2\pi(z-\zeta)} + H_\gamma(z,\zeta) \quad (32)$$

are the Green's functions with the analytic parts H_σ and H_γ.

5.1 Cascades

Consider an infinite cascade of blades with the oncoming stream parallel to the x-axis as illustrated in Fig. 3a. The cascade is characterized by the spacing (pitch) t, stagger angle β, inlet angle β_1 and outlet angle β_2 (unknown).

The periodicity boundary condition is

$$w(z) = w(z \pm k\tau), \qquad k = 1, 2, 3, \ldots, \quad (33)$$

where τ is the complex spacing

$$\tau = te^{i\beta_1}. \quad (34)$$

Using the method of images [7]

$$\begin{aligned} H_\sigma(z,\zeta) &= \frac{1}{2\tau}\left[\cot\frac{\pi(z-\zeta)}{\tau} - \frac{\tau}{\pi(z-\zeta)}\right] + \frac{i}{2\tau} \\ H_\gamma(z,\zeta) &= iH_\sigma(z,\zeta). \end{aligned} \quad (35)$$

Important limiting values are

$$\begin{aligned} &\lim_{z\to\zeta} H_\sigma(z,\zeta) = \frac{i}{2\tau}, && \lim_{x\to-\infty} H_\sigma(z,\zeta) = 0, && \lim_{x\to\infty} H_\sigma(z,\zeta) = \frac{i}{\tau}, \\ &\lim_{z\to\zeta} H_\gamma(z,\zeta) = -\frac{1}{2\tau}, && \lim_{x\to-\infty} H_\gamma(z,\zeta) = 0, && \lim_{x\to\infty} H_\gamma(z,\zeta) = -\frac{1}{\tau}. \end{aligned}$$

From Eqs. (10) and (31)–(32),

$$\lim_{x\to-\infty} w(z) = 0, \qquad \lim_{x\to\infty} w(z) = \frac{1}{\tau}(\Gamma + iQ). \quad (36)$$

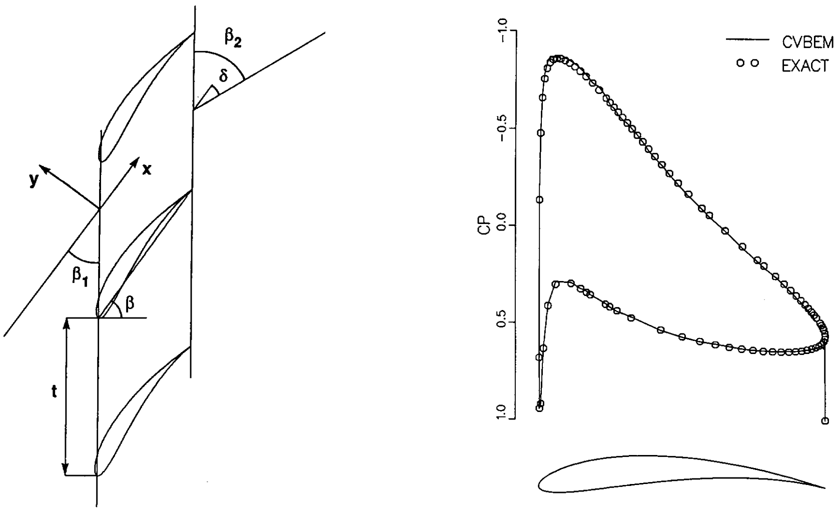
Using Eqs. (34) and (36), the deflection angle δ, measured positive in the clockwise direction, is obtained from

$$\tan\delta = \lim_{x\to\infty} \frac{\text{Im}\{w(z)\}}{1 + \text{Re}\{w(z)\}} = \frac{Q\cos\beta_1 - \Gamma\sin\beta_1}{t + \Gamma\cos\beta_1 + Q\sin\beta_1} \quad (37)$$

and the outlet angle from

$$\beta_2 = \beta_1 + \delta. \quad (38)$$

A numerical verification is given for the Gostelow compressor cascade [8]. The geometrical parameters of the cascade are: spacing to chord ratio $t/c = 0.99$, stagger angle $\beta = 52.5^o$ and inlet angle $\beta_1 = 36.5^o$, see Fig. 3a. The agreement of the CVBEM pressure distribution with the theoretical one is demonstrated in Fig. 3b. The calculated outlet angle $\beta_2 = 59.80^o$ compares reasonably well with Gostelow's exact value of 59.98^o.



(a) geometry (b) pressure

Figure 3: Gostelow cascade.

5.2 Wind tunnel walls

Flow between two parallel walls, specified by the boundary condition

$$\mathrm{Im}\{w(z)\} = 0, \qquad -\infty < x < \infty, \quad y = \pm\frac{h}{2}, \tag{39}$$

can be handled similarly. By the method of images [2]

$$H_\sigma(z,\zeta) = B(z,\zeta) + E(z,\zeta) + \frac{1}{h}, \quad H_\gamma(z,\zeta) = i[B(z,\zeta) - E(z,\zeta)], \tag{40}$$

where, using the over bar for complex conjugation,

$$\begin{aligned} B(z,\zeta) &= \frac{1}{2h}\left[\exp\left(\pi\frac{z-\zeta}{h}\right) - 1\right]^{-1} - \frac{1}{2\pi(z-\zeta)} \\ E(z,\zeta) &= -\frac{1}{2h}\left[\exp\left(\pi\frac{z-\overline{\zeta}}{h}\right) + 1\right]^{-1} \end{aligned} \tag{41}$$

From Eqs. (31)–(32) and (40)–(41) it can be shown that

$$\lim_{x\to-\infty} w(z) = 0, \qquad \lim_{x\to\infty} w(z) = \frac{1}{h}Q, \tag{42}$$

which is consistent with the ‘wake blockage’ phenomenon.

6 Concluding remarks

This paper shows that the two-dimensional vortex panel method implemented in the complex plane can be regarded as a special CVBEM case. The method is particularly well suited for potential flow problems in multiple connected domains. Its accuracy and versatility has been demonstrated on two examples.

References

[1] Hromadka II, T.V., *The Complex Variable Boundary Element Method*, Springer-Verlag: Berlin, 1984.
[2] Mokry, M., "Calculation of the Potential Flow Past Multi-Component Airfoils Using a Vortex Panel Method in the Complex Plane," LR-596, National Aeronautical Establishment, National Research Council Canada, Nov. 1978.
[3] Gakhov, F.D., *Boundary Value Problems*, Pergamon Press: Oxford, 1966.
[4] Mokry, M., Potential Flow Past Airfoils as a Riemann-Hilbert Problem, AIAA-96-2161, 1st AIAA Theoretical Fluid Mechanics Meeting, New Orleans, June 1996.
[5] Murid, A., Nasser, M., and Amin, N., A Boundary Integral Equation for the 2D External Potential Flows, to be published in *J. Appl. Mech and Eng*.
[6] Williams, B.R., "An Exact Case for the Plane Potential Flow About Two Adjacent Lifting Aerofoils," TR 71197, Royal Aircraft Establishment, Sept. 1971.
[7] Giesing, J.P., "Extension of the Douglas Neumann Program to Problems of Lifting, Infinite Cascades," LB 31653, Douglas Aircraft Co., July 1964.
[8] Gostelow, J.P., "Potential Flow through Cascades - A Comparison between Exact and Approximate Solutions," C.P. No.807, Aeronautical Research Council, 1965.