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Equivalent boundary integrals for Boussinesq–Cerruti half interaction with thin plates

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Abstract

The traditional way to include the effect of half space interaction with thin plates is to model it using domain integrals. In this paper, new sets of equivalent boundary integrals to such domain integrals are derived. The paper only covers the use of constant internal cells. Two examples are presented to show the accuracy of the present formulation.

1 Introduction

Plates on an elastic half space have many applications in civil and structural engineering. One typical application is building footings and rafts. Many simplified analytical and numerical techniques have been developed to model such problems. These models can be classified into two categories, the Winkler model and the two-parameter models. A brief summary of such models can be found in Kerr [1]. However, the most realistic modeling is the consideration of the half space as a semi-infinite problem. In this case, two analytical models can represent the half space. The first, and the most general model, is the Mindlin model [2], which utilizes complicated mathematical functions. The second model is the Boussinesq-Cerruti [3] model which can be considered as a simplified and more practical model compared to the Mindlin model.

In order to formulate the coupled problem between the plate and the half space, domain integrals are needed. Therefore, the domain needs to be discretized. An example of the domain discretization technique is the work of Hu and Hartley [4]. As the need for domain discretization loses the advantage of the boundary element method over other domain methods, transformation of such domain integrals to the boundary is desirable. However, these integrals contain the sub grade tractions which are unknown functions. Therefore, transformation of such domain integrals to the boundary is difficult. If such tractions are assumed to vary using known functions over zones in the domain (called cells), such domain integrals can be transformed to the boundary (contour) in each zone (cell). Based on this idea, Syngellakis and Bai [3] used analytical integration over plane regions to transform such domain integrals to the boundary of each cell. Paiva and Butterfield [5] followed the same idea using analytical integration in polar coordinates. However, the methods presented in Refs. [3, 5] lead to complicated kernels and are not general.

Over the past two decades, many techniques have been developed to transform domain integrals to the boundary. One of these techniques is the multiple reciprocity method (MRM) [6]. If the considered domain integral contains known functions, only the first term of the MRM is sufficient to transform such domain integral to the boundary. An example of the use of the first term of the MRM can be found in Rashed *et al.* [7]. Antes and Steinfeld [8] used an alternative technique to transform domain integrals to the boundary. The technique presented in Ref. [8] is based on Green's first identity (GFI). Rashed [9] demonstrated the relationship between the MRM and GFI.

In this paper, two new formulations are presented to transform such domain integrals to the boundary (contour). The sub grade tractions are assumed to have constant cell-wise variation. The first formulation is based on the GFI; whereas the second formulation is based on the first term of the MRM. The relevant kernels and particular solutions are derived and given in explicit form. To demonstrate the accuracy of the proposed formulation, two numerical examples are presented. The results are compared to cell integration approach and previous methods .

2 Review of the theory

In this section we present a review of the theory of thin plates resting on elastic half space. Throughout this paper, the indicial notation is used. A comma denotes differentiation ($(\cdots)_{,\theta} = \partial_\theta(\cdots) = \frac{\partial(\cdots)}{\partial x_\theta}$) and $(\cdots)_{,n}$ denotes the derivative with respect to the normal n. Greek indices will vary from 1 to 2, whereas Roman indices vary from 1 to 3.

Consider an arbitrary plate domain Ω with boundary Γ (see Figure 1). The $x_1 - x_2$ plane is assumed to be the middle surface $x_3 = 0$. The plate domain Ω is resting over and surrounded by an infinite half space boundary S.

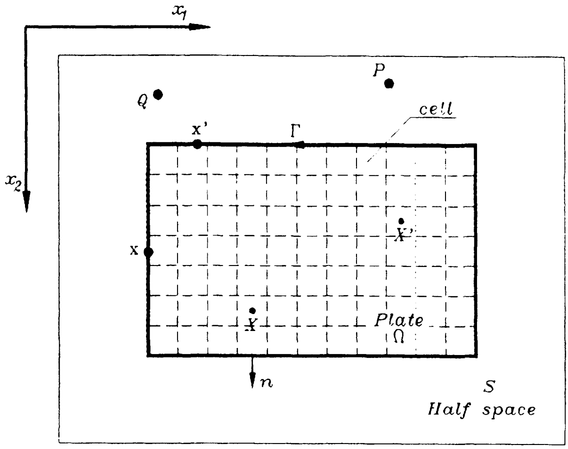

Figure 1: Plate and half space definitions.

The governing integral equations can be written as follows [3]:

The plate

$$\frac{1}{2}w(\mathbf{x}') + \int_\Gamma [V^*(\mathbf{x}',\mathbf{x})w(\mathbf{x}) - M_n^*(\mathbf{x}',\mathbf{x})w_{,n}(\mathbf{x}) - W^*(\mathbf{x}',\mathbf{x})v(\mathbf{x})$$
$$+W_{,n}^*(\mathbf{x}',\mathbf{x})m_n(\mathbf{x})]d\Gamma(\mathbf{x}) + \sum_{\text{corners}} [C^*(\mathbf{x}',\mathbf{x})w(\mathbf{x}) - W^*(\mathbf{x}',\mathbf{x})c(\mathbf{x})]$$
$$= \int_\Omega W^*(\mathbf{x}',\mathbf{X})q(\mathbf{X})d\Omega(\mathbf{X}) \qquad (1)$$

where $w(\mathbf{x}), w_{,n}(\mathbf{x}), v(\mathbf{x}), m_n(\mathbf{x}), c(\mathbf{x})$ are the plate deflection, normal slope, shear, normal bending moment and corner force respectively. The kernels with $(\cdot)^*$ denote the fundamental solution [3]. $\mathbf{x}', \mathbf{x} \in \Gamma$ are source and field points on the boundary and $\mathbf{X} \in \Omega$ is an internal field point.

The half space

$$U_i(\mathbf{P}) = \int_S U_{ij}^*(\mathbf{P},\mathbf{Q})T_j(\mathbf{Q})dS(\mathbf{Q}) \qquad (2)$$

where $\mathbf{P},\mathbf{Q} \in S$ are source and filed points respectively, U_i, T_i denote the displacement and traction components for the half space, and U_{ij}^* denotes the two point fundamental solution kernels for the Boussinesq-Cerruti half space. The expressions for U_{ij}^* are given as follows [3]:

$$U_{\alpha\beta}^* = \frac{2k}{r}[(1-\mu)\delta_{\alpha\beta} + \mu r_{,\alpha} r_{,\beta}]$$
$$U_{\alpha 3}^* = \frac{k}{r}(1-2\mu)r_{,\alpha}$$

$$U^*_{\alpha 3} = -U^*_{3\alpha}$$
$$U^*_{33} = \frac{2k(1-\mu)}{r} \quad (3)$$

in which $k = 1/(4\pi G)$, G, μ are the modulus of shear and Poisson's ratio for the half space material.

If the plate domain is discretized into cells considering the sub grade tractions T_i having constant distribution over each cell, equation (3) can be rewritten as follows:

$$U_i(\mathbf{X}') = \int_\Omega U^*_{ij}(\mathbf{X}', \mathbf{X})T_j(\mathbf{X})d\Omega(\mathbf{X}) = \sum_{\text{cells}} t_j \int_s U^*_{ij}(\mathbf{X}', \mathbf{X})ds(\mathbf{X}) \quad (4)$$

where s is the domain of each cell and t_j are constant values representing the traction components over the cell s. Equations (1) and (4) can be coupled assuming a certain material behavior and contact conditions to formulate the plate-half space interaction problem.

3 The proposed boundary integral transformation

In the present work, the domain integrals in equation (4) are considered and transformed to the boundary (contour) of each cell using two techniques: the GFI and the MRM.

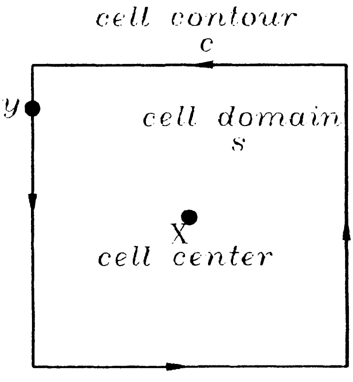

Figure 2: Typical cell domain and contour.

3.1 Using the GFI

According to the GFI [10], the considered integrals can be transformed to contour integrals as follows (consider the definitions given in Figure 2):

$$I_{ij} = \int_s U^*_{ij}(\mathbf{X}', \mathbf{X})ds(\mathbf{X}) = \int_s G^*_{ij\theta,\theta}(\mathbf{X}', \mathbf{X})ds(\mathbf{X})$$

$$= \oint_c G^*_{ij\theta}(\mathbf{X}', \mathbf{y}) n_\theta(\mathbf{y}) dc(\mathbf{y}) = \oint_c G^*_{ijn}(\mathbf{X}', \mathbf{y}) dc(\mathbf{y}) \qquad (5)$$

in which

$$G^*_{ij\theta}(\mathbf{X}', \mathbf{y}) n_\theta(\mathbf{y}) = G^*_{ijn}(\mathbf{X}', \mathbf{y}) \qquad (6)$$

and $\mathbf{X}' \notin s$ is an external collocation point. The following relationships can be easily proven:

$$\begin{aligned} (r_{,\theta})_{,\theta} &= \frac{1}{r}, \quad (\delta_{\alpha\beta} r_{,\theta})_{,\theta} = \frac{\delta_{\alpha\beta}}{r} \\ (r_{,\alpha} r_{,\theta})_{,\theta} &= \frac{r_{,\alpha}}{r}, \quad (r_{,\alpha} r_{,\beta} r_{,\theta})_{,\theta} = \frac{r_{,\alpha} r_{,\beta}}{r} \end{aligned} \qquad (7)$$

Using the above relationships, the expressions for the kernels $G^*_{ij\theta}$ can be derived as follows:

$$\begin{aligned} G^*_{\alpha\beta\theta} &= 2kr_{,\theta}[(1-\mu)\delta_{\alpha\beta} + \mu r_{,\alpha} r_{,\beta}] \\ G^*_{\alpha 3\theta} &= k(1-2\mu) r_{,\alpha} r_{,\theta} \\ G^*_{\alpha 3\theta} &= -G^*_{3\alpha\theta} \\ G^*_{33\theta} &= 2k(1-\mu) r_{,\theta} \end{aligned} \qquad (8)$$

and the expressions for the kernels G^*_{ijn} can be written as follows:

$$\begin{aligned} G^*_{\alpha\beta n} &= 2kr_{,n}[(1-\mu)\delta_{\alpha\beta} + \mu r_{,\alpha} r_{,\beta}] \\ G^*_{\alpha 3n} &= k(1-2\mu) r_{,\alpha} r_{,n} \\ G^*_{\alpha 3n} &= -G^*_{3\alpha\theta} \\ G^*_{33n} &= 2k(1-\mu) r_{,n} \end{aligned} \qquad (9)$$

3.2 Using the MRM

According to the first term of the MRM [6], the integrals I_{ij} can be transformed to boundary integrals as follows:

$$\begin{aligned} I_{ij} &= \int_s U^*_{ij}(\mathbf{X}', \mathbf{X}) ds(\mathbf{X}) \\ &= \int_s \nabla^2 V^*_{ij}(\mathbf{X}', \mathbf{X}) ds(\mathbf{X}) = \oint_c V^*_{ij,n}(\mathbf{X}', \mathbf{y}) n_\theta(\mathbf{y}) dc(\mathbf{y}) \end{aligned} \qquad (10)$$

where ∇^2 is the two-dimensional Laplace operator. The following relationships can be easily proven:

$$\begin{aligned} \nabla^2\left(\frac{r}{3}(2\delta_{\alpha\beta} - r_{,\alpha} r_{,\beta}\right) &= \frac{r_{,\alpha} r_{,\beta}}{r} \\ \nabla^2\left(\frac{r^2}{4}\delta_{\alpha\beta}\right) &= \delta_{\alpha\beta} \\ \nabla^2\left(\frac{r_{,\alpha} r \ln r}{2}\right) &= \frac{r_{,\alpha}}{r} \\ \nabla^2(r) &= \frac{1}{r} \end{aligned} \qquad (11)$$

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Using the above relationships, the expressions for the kernels V_{ij}^* can be derived as follows:

$$\begin{aligned} V_{\alpha\beta}^* &= \frac{2k}{3}\{(3-\mu)r\delta_{\alpha\beta} - \mu r r_{,\alpha} r_{,\beta}\} \\ V_{\alpha 3}^* &= \frac{k(1-2\mu)}{2} r r_{,\alpha} \ln r \\ V_{\alpha 3}^* &= -V_{3\alpha}^* \\ V_{33}^* &= 2k(1-\mu)r \end{aligned} \quad (12)$$

and their normal derivatives can be written as follows:

$$\begin{aligned} V_{\alpha\beta,n}^* &= \frac{2k}{3}\{\mu[r_{,\alpha}r_{,\beta}r_{,n} - r_{,\alpha}n\beta - r_{,\beta}n\alpha] + (3-\mu)\delta_{\alpha\beta}r_{,n}\} \\ V_{\alpha 3,n}^* &= \frac{k(1-2\mu)}{2}(n\alpha \ln r + r_{,\alpha}r_{,n}) \\ V_{\alpha 3,n}^* &= -V_{3\alpha,n}^* \\ V_{33,n}^* &= 2k(1-\mu)r_{,n} \end{aligned} \quad (13)$$

3.3 Equivalence of the GFI to the MRM

In order to demonstrate the equivalence between the GFI formulation and the MRM formulation, the following kernels are computed:

$$S_{ij,\theta}^* = V_{ij,\theta}^* - G_{ij\theta}^* \quad (14)$$

and the expressions for the kernels $S_{ij,\theta}^*$ can be obtained as follows:

$$\begin{aligned} S_{\alpha\beta,\theta}^* &= \frac{2k}{3}\{\mu[-2r_{,\alpha}r_{,\beta}r_{,\theta} - r_{,\alpha}\delta_{\theta\beta} - r_{,\beta}\delta_{\theta\alpha}] + 2\mu\delta_{\alpha\beta}r_{,\theta}\} \\ S_{\alpha 3,\theta}^* &= \frac{k(1-2\mu)}{2}(\delta_{\theta\alpha}\ln r - r_{,\alpha}r_{,\theta}) \\ S_{\alpha 3,\theta}^* &= -S_{3\alpha,\theta}^* \\ S_{33,\theta}^* &= 0 \end{aligned} \quad (15)$$

It is easy to show that $S_{ij,\beta\beta}^* = \nabla^2 S_{ij}^* = 0$, which verifies that S_{ij}^* are harmonic functions [10]. In other words, the following integral identity can be written using the MRM:

$$\int_s S_{ij,\beta\beta}^* ds = \oint_c S_{ij,n}^* dc = 0 \quad (16)$$

Therefore, the boundary integral of the kernels $V_{ij,n}^*$ is equivalent to the boundary integral of the kernel G_{ijn}^*.

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4 Self collocation

Due to the consideration of constant internal cells only, two types of collocation positions are considered; i.e., external and internal. The kernels derived in the present work for the equivalent contour integrals are obtained assuming that the collocation point $\mathbf{X}'$ is an external collocation point. When the point $\mathbf{X}'$ is moved to have the same location as the point $\mathbf{X}$ (in the case of the self (internal) collocation), equations (5) and (10) are also valid for this case as the derived kernels in the contour integrals I_{ij} are not strongly singular.

5 Numerical examples

In this section, two numerical examples are presented to demonstrate the accuracy of the present formulations. The numerical integrations are performed using the Gauss-Legendre scheme. The non-linear coordinate transformation presented in Ref. [11] is used to compute weakly singular kernels in the domain and along the boundary. In all examples, Poisson's ratio is taken to be 0.3 and the value of k is taken to be unity.

5.1 Example 1: Plate with 4 square cells

In this example, the 4×4 plate shown in Figure 3 is considered. The plate is discretized as shown in 4 square cells. Table 1 shows a comparison of the results for the value of the integrals I_{ij} computed at the collocation point shown in Figure 3. It can be seen that the results of the GFI and the MRM are in good agreement with the results of Ref. [3]. Also, it can be seen that the accuracy of computing boundary (contour) integrals is better than that of domain integrals, even when using only 4 Gauss points.

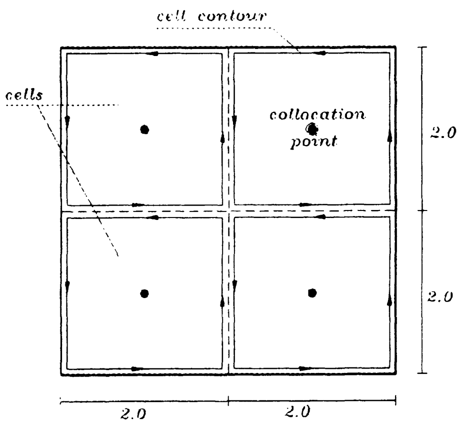

Figure 3: Plate with 4 square cells.

Table 1: Results of example 1: Plate with 4 square cells.

Integral	Cell-4*	Cell-10*	Cell-40*	MRM-4*	GFI-4*	Ref[1]-4*
I_{11}	9.77259	10.34196	10.64814	10.75470	10.75470	10.75470
I_{12}	0.20031	0.20031	0.20031	0.20031	0.20031	0.20031
I_{22}	9.77259	10.34196	10.64814	10.75470	10.75470	10.75470
I_{13}	-0.59461	-0.59468	-0.59468	-0.59468	-0.59468	-0.59468
I_{23}	-0.59461	-0.59468	-0.59468	-0.59468	-0.59468	-0.59468
I_{33}	8.04802	8.51691	8.76906	8.85681	8.85681	8.85681

* Numbers denote the used number of Gauss points

5.2 Example 2: Irregular cell

In order to demonstrate the generality of the present formulations, the irregular cell shown in Figure 4 is considered. Two collocation points are considered, external and internal (see Figure 4). Tables 2 and 3 give a comparison of the results for the values of the integrals I_{ij} for external and internal collocations, respectively. It can be seen that results of the present formulations are in good agreement with the results of Ref. [3]. Also, it can be seen that cell integration is not accurate when using 4 Gauss points, especially in the case of internal collocation. Figures 5 and 6 show the computed values of the integrals I_{ij} using cell integration with different numbers of Gauss points as internal collocation points. It can be seen that the accuracy of the cell integration improved by increasing the number of Gauss points and these values converge to the computed values using boundary (contour) integrals with only 4 Gauss points.

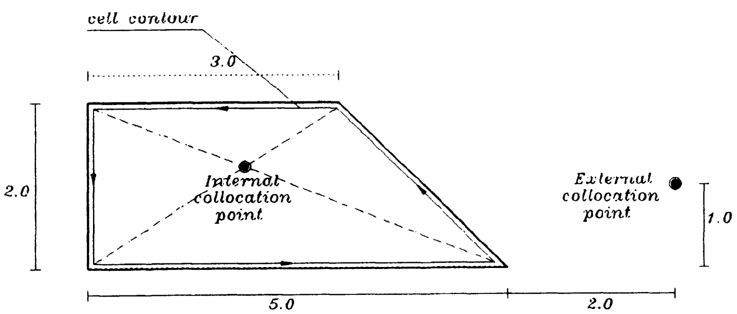

Figure 4: Irregular cell.

Table 2: Results of example 2: Irregular cell, external collocation.

Integral	Cell-4*	Cell-10*	MRM-4*	GFI-4*	Ref[1]-4*
I_{11}	3.40763	3.40765	3.40765	3.40765	3.40765
I_{12}	0.04249	0.04250	0.04250	0.04250	0.04250
I_{22}	2.42140	2.42144	2.42144	2.42144	2.42144
I_{13}	-0.67859	-0.67859	-0.67859	-0.67859	-0.67859
I_{23}	-0.02941	-0.02942	-0.02942	-0.02942	-0.02942
I_{33}	2.40019	2.40021	2.40021	2.40021	2.40021

* Numbers denote the used number of Gauss points

Table 3: Results of example 2: Irregular cell, internal collocation.

Integral	Cell-4*	Cell-200*	MRM-4*	GFI-4*	Ref[1]-4*
I_{11}	14.08005	16.49526	16.54993	16.54993	16.54993
I_{12}	0.24205	-0.07655	-0.08713	-0.08713	-0.08713
I_{22}	12.29691	15.42225	15.48981	15.48981	15.48981
I_{13}	0.03635	0.05969	0.05904	0.05904	0.05904
I_{23}	-0.44817	-0.49592	-0.49566	-0.49566	-0.49566
I_{33}	10.86110	13.14251	13.19283	13.19283	13.19283

* Numbers denote the used number of Gauss points

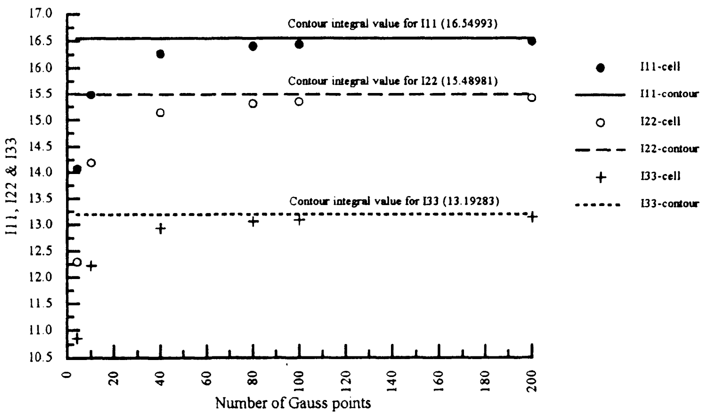

Figure 5: Irregular cells, results for the integrals I_{11}, I_{22} and I_{33}.

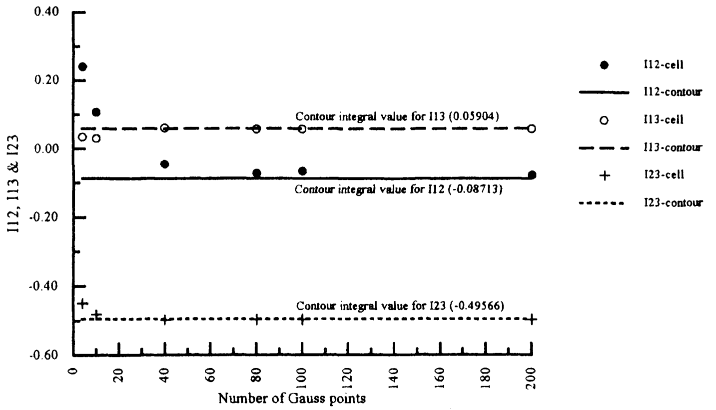

Figure 6: Irregular cells, results for the integrals I_{12}, I_{13} and I_{23}.

6 Conclusions

In this paper, equivalent boundary (contour) integrals were derived for the elastic half space interaction effect for thin plates. The formulations were based on assuming a cell-wise constant traction distribution over the plate domain. Two formulations were derived: the first employed Green's first identity and the second employed the first term of the multiple reciprocity method. The following conclusions may be drawn from the present work:

1. For a certain domain integral, both Green's first identity and the multiple reciprocity method can be used to transform it to the boundary. However, Green's first identity leads to simpler equivalent boundary integrals than that of the multiple reciprocity method.

2. Both kernels derived based on the GFI and the MRM have the same numerical accuracy.

3. Evaluation of contour integrals is more accurate than evaluation of domain integrals.

The present formulations can be easily extended to linear and quadratic subgrade traction distribution over the plate subdomains.

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