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Green's functions in the numerical solution of some inverse boundary value problems

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Abstract

We use boundary elements, with least squares optimization, to solve identification problems in electrostatics and steady state heat conduction in conductive plates with simple shapes, e.g. the circular wedge. The plate has a hole in it, with a flux across the hole boundary. Voltage or heat is applied to part of the plate edge; the remainder of the plate edge is grounded or insulated. For the inverse problem, an additional voltage or heat measurement is taken on an insulated part of the plate edge; from this extra measurement we (1) determine the structure of the hole, given the flux across its boundary, or (2) given the hole, determine the flux across its boundary. Special boundary value properties of Green's functions are exploited in this numerical treatment of the identification problem.

1 Introduction

Numerical methods for solution of inverse boundary value problems of internal cavity detection or flux reconstruction in planar domains have been obtained before[1,2,9,10]. These methods use least squares optimization, coupled with an integral equation for construction of a potential at the boundary as part of the iteration process. Our method applies the same idea, except that Green's functions are used at the potential construction stage. Green's functions or matrices for elliptic partial differential equations with mixed boundary conditions exist in closed form for a number of domains of a variety of simple shapes. Melnikov[4] has developed techniques which make it possible to obtain representations for these Green's functions. Melnikov's Green's functions have been applied to several problems in applied mechanics[4,5,7], and this paper, a continuation of our previous work[6], is an extension of their applications to solution of inverse boundary value problems.

Suppose that a domain is occupied by an electrically or thermally conductive material which contains a hole. From electrostatic or steady state thermal measurements at the boundary of the domain, we would like to characterize the location, size, and shape of the hole, or some other unknown property such as current flux across the hole boundary. We assume that voltage or heat satisfies Laplace's equation inside of the domain, and formulate the problem of finding the potential as an integral equation on the hole boundary. However, the hole or its flux is unknown, so an initial estimate is required; the boundary integral equation is solved for the corresponding potential; this potential is compared with the data for the problem, and the estimate is updated by means of the least squares method, with the object of minimizing the difference between measured data and data corresponding to the estimated hole or flux. Use of a Green's function situates the integral equation on the flaw boundary; for each updated data estimate, maximum error is confined to the boundary of the guessed cavity, and accuracy is achieved in the simulated data at the plate boundary.

2 Formulation of Problem

Let Ω be a domain in the complex plane $\mathbf{C}$, with piecewise smooth boundary

$$\Gamma = \bigcup_{i=1}^{m} \Gamma_i$$

We assume that an electrically or thermally conductive material occupies Ω, and that its constant conductivity equals 1. The boundary value problem for voltage or temperature u in Ω is

$$\Delta u = 0 \qquad z \in \Omega \tag{1}$$

$$\alpha_i \frac{\partial u}{\partial n_i} + \beta_i u = h_i \qquad z \in \Gamma_i, \quad i = 1, \ldots, m \tag{2}$$

where Δ is Laplace's operator, α_i, β_i are known functions not simultaneously zero on Γ_i, n_i are unit normal vectors exterior to Ω, and h_i are functions given only on Γ_i. We consider the case when $\alpha_i = 0$, $\beta_i = 1$ on some Γ_i, while $\alpha_i = 1$, $\beta_i = 0$ on the remaining pieces of Γ.

Let $G(z,t)$ be the Green's function which satisfies the boundary conditions (2) on Γ with $h_i = 0$ for all $i = 1, \ldots, m$ and

$$-\Delta_z G(z,t) = \delta(z-t), \qquad t, z \in \Omega. \tag{3}$$

$G(z,t)$ tends to $(-1/2\pi)\ln|z-t|$ as z tends to t.

Suppose that there is a hole D, with boundary Γ_0, inside of Ω, and that $\overline{D} \subset \Omega$. For the numerical solution of our inverse problem, the recovery of an interior hole from measurements on Γ, we will need numerical solutions of direct problems

$$\Delta u \quad = \quad 0 \qquad z \in \Omega \setminus D \tag{4}$$

with boundary data (2), and

$$\frac{\partial u}{\partial n_0} \quad = \quad g \qquad z \in \Gamma_0 \tag{5}$$

where possibly $g \equiv 0$. These solutions may be constructed by means of a Fredholm boundary value problem obtained from the jump condition[8] on Γ_0 for the function

$$\Psi(z) = \frac{1}{2\pi i}\int_{\Gamma_0} \frac{\psi(t)}{t-z}dt \qquad z \in \Omega \setminus D,$$

$$\Psi^-(z) = -\frac{1}{2}\psi(z) + \frac{1}{2\pi i}\int_{\Gamma_0} \frac{\psi(t)}{t-z}dt \qquad z \in \Gamma_0$$

where Ψ^- is the limit of Ψ at Γ_0 from inside $\Omega \setminus D$, in the following way: First, the Green's function (3) is used to obtain the solution w to (1), (2) in Ω without the hole,

$$w(z) = \int_{\Gamma\setminus\{\Gamma_i|\alpha_i=0\}} G(z,t)h_i(t)d\Gamma_t - \int_{\Gamma\setminus\{\Gamma_i|\beta_i=0\}} \frac{\partial}{\partial n}G(z,t)h_i(t)d\Gamma_t \tag{6}$$

Then u is found in the form $u = v + w$, where v is the harmonic function in $\Omega \setminus D$ with homogeneous data on Γ obtained by letting

$$v(z) = \int_{\Gamma_0} G(z,t)\mu(t)d\Gamma_t \tag{7}$$

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and solving for the density μ in the equation obtained from the jump condition,

$$\mu(z) - 2\int_{\Gamma_0} \frac{\partial}{\partial n_0} G(z,t)\mu(t)d\Gamma_t = -2[g(z) - \frac{\partial}{\partial n_0} w(z)], \quad z \in \Gamma_0 \quad (8)$$

where the normal derivative is with respect to z. Numerical solution can be obtained by means of a matrix equation for μ at points on Γ_0.

Now suppose that the region Ω contains a hole D, and that the location, size, and shape of D are unknown. The inverse hole determination problem is the following: Given (4), (2), and (5), and $g = 0$ on the boundary of unknown D, determine D from one additional measurement on an accessible part Γ_i of the boundary Γ. This additional measurement will be $u = f$ where $\beta_i = 0$, or $\partial u/\partial n_i = f$ if $\alpha_i = 0$. Note that if D were known and this additional measurement were given, then the problem of determining u would be ill-posed.

For the inverse flux determination problem, suppose that all properties of the hole D are known, and (4), (2) are given with all $h_i = 0$. Then given the additional measurement on Γ, determine flux g on the boundary Γ_0 of D as a function of some parameter, say of arclength on Γ_0, or of a radial variable.

3 Solution of Inverse Problem

The following numerical method for solution of the inverse problems combines the boundary element method for solution of the direct problem with a least squares optimization procedure[3] for updated approximations.

The method requires an initial estimate for D, which we call $\tilde{D}$ with boundary $\tilde{\Gamma}_0$; or in the case of flux determination, an initial estimate $\tilde{g}$ for unknown g. Solution of the resulting integral equation (8) for $\tilde{\mu}$ on $\tilde{\Gamma}_0$, and insertion in (7) with $\Gamma_0 = \tilde{\Gamma}_0$, will result in calculated data $\tilde{f}$ on Γ_i, to be compared with data f, then updated by the least squares optimization scheme.

This update is achieved by means of the objective functional

$$F(\tilde{f}(\mathbf{c}), f) = \int_{\Gamma_i} |\tilde{f}(t;\mathbf{c}) - f(t)|^2 d\Gamma_t$$

where $\mathbf{c}$ is a set of parameters which characterize $\tilde{\Gamma}_0$ or $\tilde{g}$. The following problem is equivalent to the solution of the inverse problem:

Minimize the difference F with respect to $\tilde{D}$ or $\tilde{g}$. This may be done conveniently using some parameterization of the estimate $\tilde{\Gamma}_0$, $\tilde{t}(\sigma) = \tilde{t}(\sigma;\mathbf{c})$, where σ will become the variable of integration in (7) and (8). We will also use $\tilde{z}(\theta) = \tilde{z}(\theta;\mathbf{c})$ to denote the same parameterization. In the flux problem, $\tilde{g}(\sigma) = \tilde{g}(\sigma;\mathbf{c})$, with parameterization $t(\sigma)$ given.

To use the optimization procedure[3], we assume that $\mathbf{c}+\delta$, gives the closest parameterization of the unknown, where the first order Taylor approximation of F is

$$\Phi(\mathbf{c}+\delta) = \int_{\tilde{\Gamma}_i} [\tilde{f}(t;\mathbf{c}) + \sum_{j=1}^{n} \frac{\partial \tilde{f}}{\partial c_j}(t;\mathbf{c})\delta_j - f(t)]^2 d\Gamma_t$$

Then the assumption

$$\frac{\partial \Phi}{\partial \delta_k}(\mathbf{c}+\delta) = 2\int_{\tilde{\Gamma}_i} [\tilde{f}(t;\mathbf{c}) + \sum_{j=1}^{n} \frac{\partial \tilde{f}}{\partial c_j}(t;\mathbf{c})\delta_j - f(t)]\frac{\partial \tilde{f}}{\partial c_k}(t;\mathbf{c}) d\Gamma_t = 0$$

for $k = 1,\ldots,n$ results in the matrix equation

$$\mathbf{A}\delta = \mathbf{q}$$

for δ. The $n \times n$ matrix $\mathbf{A} = (a_{kj})$ and $\mathbf{q} = (q_1,\ldots,q_n)$ have entries

$$a_{kj} = \int_{\tilde{\Gamma}_i} \frac{\partial \tilde{f}}{\partial c_j}(t;\mathbf{c}) \frac{\partial \tilde{f}}{\partial c_j}(t;\mathbf{c}) d\Gamma_t$$

and

$$q_k = \int_{\tilde{\Gamma}_i} [\tilde{f}(t;\mathbf{c}) - f(t)] \frac{\partial \tilde{f}}{\partial c_j}(t;\mathbf{c}) d\Gamma_t$$

respectively.

The derivatives $\partial \tilde{f}/\partial c_j$, for $z \in \Gamma_i$, are calculated on the parameterization of the integral (7), with $\tilde{t}$ given in case g is to determined,

$$\tilde{u}(z) = \int_0^{2\pi} G(z,\tilde{t}(\sigma))\tilde{\mu}(\sigma)|\tilde{t}'(\sigma)| d\sigma$$

as follows:

$$\frac{\partial \tilde{u}}{\partial c_j}(z) = \frac{\partial \tilde{v}}{\partial c_j}(z) = \int_0^{2\pi} G(z,\tilde{t}(\sigma))\frac{\partial \tilde{\mu}}{\partial c_j}(\sigma)|\tilde{t}'(\sigma)| d\sigma + \int_0^{2\pi} \frac{\partial}{\partial c_j}[G(z,\tilde{t}(\sigma))|\tilde{t}'(\sigma)|]\tilde{\mu}(\sigma) d\sigma$$

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where the derivatives $\partial\tilde{\mu}/\partial c_j$ are calculated from integral equations derived from (8), for $0 \leq \theta \leq 2\pi$

$$\frac{\partial\tilde{\mu}}{\partial c_j}(\theta) - 2\int_0^{2\pi} \frac{\partial}{\partial n_0} G(\tilde{z}(\theta), \tilde{t}(\sigma)) \frac{\partial\tilde{\mu}}{\partial c_j}(\sigma)|\tilde{t}'(\sigma)|d\sigma = 2\int_0^{2\pi} \frac{\partial}{\partial c_j}[\frac{\partial}{\partial n_0} G(\tilde{z}(\theta), \tilde{t}(\sigma))|\tilde{t}'(\sigma)|]\tilde{\mu}(\sigma)d\sigma - 2\frac{\partial}{\partial c_j}[\tilde{g}(\tilde{z}(\theta)) - \frac{\partial}{\partial n_0} w(\tilde{z}(\theta))], \tag{9}$$

where the $\partial\tilde{g}/\partial c_j$ terms are zero if D is to be determined; the parameter derivatives of $\partial w/\partial n_0$ are zero if D is known and g is to be determined.

We solve for δ by means of the corrected Gauss–Newton scheme[3]. $\mathbf{c}$ is updated by $\mathbf{c} = \mathbf{c} + \delta$. The procedure is repeated until a convergence criterion is met, say the difference of successive approximations ≈ 0. The output $\mathbf{c}$ will lead to the parameterization of the approximate solution to the inverse problem.

4 Numerical Example

Let Ω be the right circular sector of radius R, $\Omega = \{z = re^{i\theta} : 0 \leq r \leq R, 0 \leq \theta \leq \pi/2\}$. The boundary Γ consists of Γ_1, the circular part; Γ_2, the interval $[0, R]$ on the x-axis; and Γ_3, the interval $[0, R]$ on the y-axis. The Green's function for Ω with mixed boundary conditions

$$\frac{\partial u}{\partial r}|_{\Gamma_1} = u|_{\Gamma_2} = \frac{\partial u}{\partial \theta}|_{\Gamma_3} = 0 \tag{10}$$

is

$$G(z, \xi) = \frac{1}{2}\ln\frac{|z - \bar{\xi}||z + \xi||R^2 - z\xi||R^2 + z\bar{\xi}|}{|z - \xi||z + \bar{\xi}||R^2 - z\bar{\xi}||R^2 + z\xi|}$$

For the inverse problem of determining the hole D, we suppose that the flux $\partial u/\partial n_0 = 0$ on Γ_0. For our simulation, we apply voltage of the form $h_2(z) = 1 - \cos 2\pi/R$ on Γ_2. The resulting voltage f on Γ_3 is measured, and the method of section 3 applied to determine D.

Figure 1 depicts the eighth iteration of approximation of a nonconvex hole (solid line) by an ellipse; the approximation is from exact data; the initial estimate is circular. Figure 2 shows exact data

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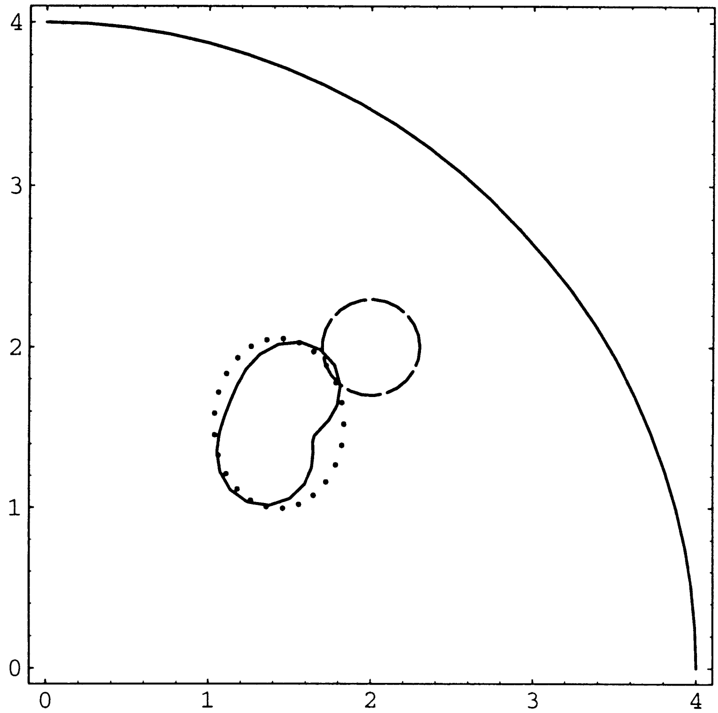
along with data containing five percent random additive noise; at the twelfth iteration, the reconstruction (figure 3) begins to move away from the true hole, which indicates the need for stabilization of the algorithm.

Figure 4 shows results of the simulation of the flux determination problem. We suppose that the data satisfies (10), and measure $u = f$ on Γ_3. D is assumed to be a disk of radius 1/2 centered at (3/2, 3/2), and the true flux (solid line) is the polynomial $g(x) = (0.01)x^2(x-2)(x-3)(x-5)(x-2\pi)^2$. The initial estimate is the dashed line. We try to approximate g with the trigonometric sum

$$\tilde{g}(\theta) = \frac{1}{2}a_0 + \sum_{k=1}^{5}(a_k \cos k\theta + b_k \sin k\theta)$$

with respect to the parameters a_k, b_k.



. Figure 1: Approximation of hole

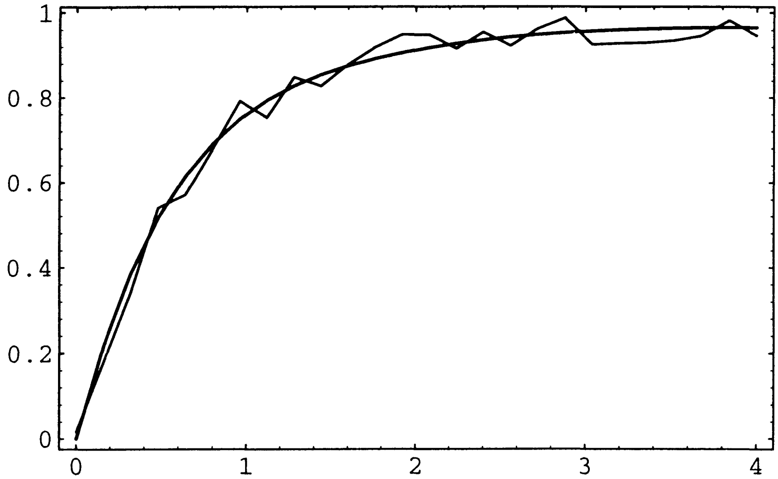


Figure 2: Noisy data for hole

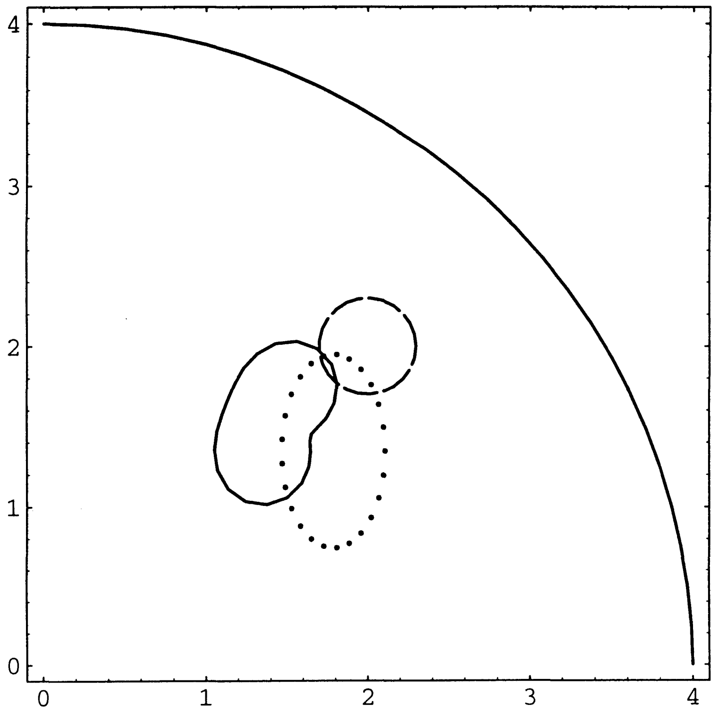


Figure 3: Noisy approximation of hole

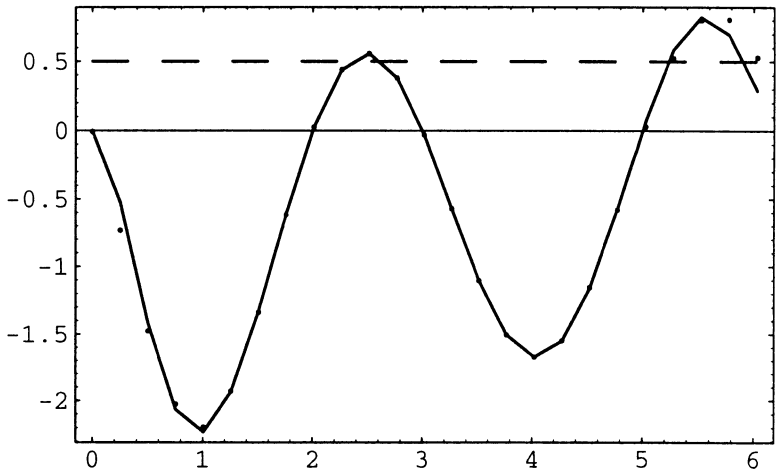


Figure 4: Approximation of flux

Conclusion

The algorithm for solution of the inverse problem is very simple at the data simulation stage due to special boundary properties of the Green's function. This data simulation can occur a number of times per iteration, due to the nature of the least squares updating process, so it is an advantage to have as simple a formulation as possible.

The Green's function algorithm can be implemented in any domain for which a closed form representation of the Green's function exists[4]. We have results for the infinite strip and the semistrip. Ongoing work includes stabilization of the algorithm in the presence of noisy data. We intend to obtain a more accurate reconstruction of the nonconvex hole in figure 1 by means of a spline approximation scheme. We also plan to combine reconstruction procedures for the hole and flux, and try to compute hole and flux simultaneously from additional measurements at the plate boundary.

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