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The quite general BEM for strongly non-linear problems

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Abstract

In this paper, the basic ideas described in [4] is greatly generalized to give a kind of high-order BEM formulations for strongly non-linear problems governed by quite general differential operators which may NOT contain any linear operators at all. As a result, the traditional BEM which treats the non-linear terms as the inhomogeneities is only a special case of the proposed BEM. One simple example is used to illustrate its effectiveness

1 Introduction

Although BEM is in principle based on the linear superposition of fundamental solutions, many researchers have applied it to solve non-linear boundary-value problems governed by the non-linear differential operator $\mathcal{A}$:

$$\mathcal{A}(u) = f(\vec{r}). \tag{1}$$

If this non-linear operator $\mathcal{A}$ can be divided into two parts $\mathcal{L}_0$ and $\mathcal{N}_0$, where, $\mathcal{L}_0$ is linear, $\mathcal{N}_0$ is non-linear and $\mathcal{A} = \mathcal{L}_0 + \mathcal{N}_0$ holds. Then traditionally, writing the original equation (1) as $\mathcal{L}_0(u) = f - \mathcal{N}_0(u)$, we can obtain the following equation of integral operator

$$c(\vec{r})u(\vec{r}) = \int_\Gamma [u\ \mathcal{B}_0(\omega_0) - \omega_0\ \mathcal{B}_0(u)]\, d\,\Gamma + \int_\Omega [f - \mathcal{N}_0(u)]\ \omega_0 d\,\Omega \tag{2}$$

where, ω_0 is the fundamental solution of the adjoint operator of the linear differential operator $\mathcal{L}_0$, $\mathcal{B}_0$ is its boundary operator, Γ denotes the boundary of the domain Ω. Note that the domain integral of above equation contains the unknown function $u(\vec{r})$ so that iteration is necessary.

This traditional BEM described above has the following restrictions:

1. Many non-linear differential operators do NOT contain any linear operators at all, i.e., $\mathcal{A} = \mathcal{L}_0 + \mathcal{N}_0$ does NOT hold, so that the traditional BEM is useless in this case.

2. Even if non-linear differential operator $\mathcal{A}$ contains such a linear operator $\mathcal{L}_0$, $\mathcal{L}_0$ may be so complex that its fundamental solution is either unknown or unfamiliar to us.

So, it seems necessary to develop a kind of new BEM for quite general non-linear problems,

(I) which should be suited to equations governed by quite general non-linear differential operators which may NOT contain any linear operators at all;

(II) which should give us great freedom to select a proper linear operator whose fundamental solution is familiar to us, no matter whether the linear operator $\mathcal{L}_0$ exists or not;

(III) which should contain logically the traditional BEM.

In this paper, the basic ideas described in [4] are greatly generalized to give a new quite general BEM for strongly non-linear problems. One simple example is used to illustrate the effectiveness of it.

2 The Basic Ideas of the Proposed BEM

We consider again the equation (1). Selecting a proper linear operator $\mathcal{L}$ whose fundamental solution is familiar to us, we construct a homotopy $v(\vec{r}, p) : \Omega \times [0,1] \to \mathbf{R}$, which satisfies

$$\mathcal{L}(v) = (1-p)\,\mathcal{L}(u_0) + p\,\left[\mathcal{L}(v) - \mathcal{A}(v) + f\right] \qquad , \qquad p \in [0,1], \qquad (3)$$

where, $u_0(\vec{r})$ is a free selected initial solution, $p \in [0,1]$ is the imbedding parameter, $v(\vec{r}, p)$ is now a function of both $p \in [0,1]$ and $\vec{r} \in \Omega$. We call the equation (3) the *zero-order deformation equation.*

Obviously, from equation (3), the following two expressions

$$\begin{aligned} v(\vec{r}, 0) &= u_0(\vec{r}), & (4)\\ v(\vec{r}, 1) &= u(\vec{r}), & (5) \end{aligned}$$

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hold, where $u(\vec{r})$ is the solution of equation (1). Therefore, $u_0(\vec{r})$ and $u(\vec{r})$ are homotopic, denoted as $v(\vec{r},p) : u_0(\vec{r}) \simeq u(\vec{r})$. Assume that the "continuous deformation" $v(\vec{r},p)$ is smooth enough about p so that

$$v^{[m]}(\vec{r},p) = \frac{\partial^m v(\vec{r},p)}{\partial p^m} \quad , \qquad m = 1,2,3,\cdots, \tag{6}$$

called *mth-order deformation derivatives*, exist. Then, according to the theory of Taylor's series, we have from (4) that

$$\begin{aligned} v(\vec{r},p) &= v(\vec{r},0) + \sum_{m=1}^{\infty} \left.\frac{\partial^m v(\vec{r},p)}{\partial p^m}\right|_{p=0} \left(\frac{p^m}{m\,!}\right) \\ &= u_0(\vec{r}) + \sum_{m=1}^{\infty} \left(\frac{p^m}{m\,!}\right) v_0^{[m]}(\vec{r}), \end{aligned} \tag{7}$$

where, $v_0^{[m]}(\vec{r})$ is the value of $v^{[m]}(\vec{r},p)$ at $p = 0$, which can be obtained in the way described later. For simplicity, we call above expression *Taylor's homotopy series.* The value of the convergence radius ρ of the serie (7) is generally finite. So, in case $\rho \leq 1$, we have

$$v(\vec{r},\lambda) = u_0(\vec{r}) + \sum_{m=1}^{\infty} \left[\frac{v_0^{[m]}(\vec{r})}{m\,!}\right] \lambda^m, \tag{8}$$

where, $0 < \lambda \leq \rho \leq 1$. Note that $v(\vec{r},\lambda)$ obtained by above expression is mostly a better approximation than the initial solution $u_0(\vec{r})$ so that expression (8) gives a family of the high-order iterative formulations:

$$u_{k+1}(\vec{r}) = u_k(\vec{r}) + \sum_{m=1}^{M} \left[\frac{v_0^{[m]}(\vec{r})}{m\,!}\right] \lambda^m, \qquad (k = 0,1,2,\cdots), \tag{9}$$

Differentiating the zero-order deformation equation (3) m times with respect to the imbedding parameter p and then setting $p = 0$, we obtain the *mth-order deformation equations* at $p = 0$ as follows:

$$\mathcal{L}(v_0^{[m]}) = f_m(\vec{r}) \qquad (m \geq 1), \tag{10}$$

where

$$f_1(\vec{r}) = f - \mathcal{A}(u_k), \tag{11}$$

$$f_m(\vec{r}) = m \left\{ \mathcal{L}(v_0^{[m-1]}) - \left.\frac{d^{m-1}\mathcal{A}(v)}{dp^{m-1}}\right|_{p=0} \right\} \qquad (m > 1). \tag{12}$$

Note that $f_1(\vec{r})$ is the minus residual of the original equation (1) and is the same for ANY linear operators $\mathcal{L}$ suited to the proposed BEM.

Then, according to (2), we have the corresponding boundary integral equation

$$c(\vec{r})v_0^{[m]}(\vec{r}) = \int_\Gamma \left[v_0^{[m]}\,\mathcal{B}(\omega) - \omega\mathcal{B}(v_0^{[m]})\right] d\,\Gamma + \int_\Omega f_m\;\omega\; d\,\Omega. \qquad (13)$$

Note that we have now great freedom to select a simple, proper linear operator $\mathcal{L}$ whose fundamental solution ω is famaliar to us, no matter whether the considered non-linear problem contains linear operators or not! Especially, if $\mathcal{A} = \mathcal{L}_0 + \mathcal{N}_0$ holds and we select $\mathcal{L}_0$ as the linear operator, the formulation (13) in case $m = 1$ gives the same expression as the expression (2). Therefore, the three demands (I) (II) (III) listed in the first section are completely satisfied.

3 One Simple Numerical Example

In order to illustrate the effectiveness of the proposed BEM, let us consider such a non-linear boundary-value problem which does NOT contain any linear operators at all :

$$\begin{aligned} 2e^{U_{xx}}\sin(U_{xx}) + \alpha\left(U^2 + U_x^2\right) &= \alpha - e^{-\sin(x)}\sin(\sin(x)) \\ & \quad x \in [0, 2\pi], \alpha \geq 0, \end{aligned} \qquad (14)$$

with the two boundary conditions $U(0) = U(2\pi) = 0$.

We use respectively the following two sorts of linear operators

$$\text{MODE 1} : \mathcal{L}_1(U) = U_{xx} - \beta^2 U \qquad \beta > 0\;,$$

$$\text{MODE 2} : \mathcal{L}_2(U) = U_{xx} + \beta^2 U \qquad \beta > 0\;,$$

to construct the corresponding zero-order deformation equation

$$\begin{aligned} \mathcal{L}_\gamma(V) &= (1-p)\mathcal{L}_\gamma(U_0) + p\left\{\mathcal{L}_\gamma(V) - \mathcal{A}(V) + f\right\}, \\ & \quad x \in [0, 2\pi] \qquad p \in [0,1] \qquad (\gamma = 1, 2) \end{aligned} \qquad (15)$$

which has the two boundary conditions $V(0,p) = V(2\pi, p) = 0$, where, $\mathcal{A}(V) = 2e^{V_{xx}}\sin(V_{xx}) + \alpha\left(V^2 + V_x^2\right), f(x) = \alpha - e^{-\sin(x)}\sin(\sin(x))$,and $V(x,p) : [0, 2\pi] \times [0, 1] \to \mathbf{R}$ is a kind of homotopy, $U_0(x)$ is a selected initial solution satisfying the boundary conditions $U_0(0) = U_0(2\pi) = 0$.

Similarly, we can obtain the corresponding high-order iterative formulations

$$U_{k+1}(x) = U_k(x) + \sum_{m=1}^{M} \frac{\lambda^m V_0^{[m]}(x)}{m!}, \qquad (k = 0, 1, 2, \cdots), \qquad (16)$$

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Table 1: iterative times of example 1 for different α

α	λ	MODE 1	MODE 2
0	0.25	14	11
0.25	0.20	9	21
1.00	0.10	19	43
5.00	0.025	45	59
100	0.00125	233	228
1000	0.000125	356	350

where, $V_0^{[m]}(x)$ satisfies the linear differential equation

$$\mathcal{L}_\gamma(V_0^{[m]}) = f_m(x), \qquad x \in [0, 2\pi] \qquad (\gamma = 1, 2, \;\; m = 1, 2, 3 \cdots) \tag{17}$$

with the two boundary conditions $V_0^{[m]}(0) = V_0^{[m]}(2\pi) = 0$. Here,

$$f_1(x) = \alpha - e^{-\sin(x)} \sin(\sin(x)) - \mathcal{A}(U_k), \tag{18}$$

$$f_2(x) = 2\left\{\mathcal{L}_\gamma(V_0^{[1]}) - 2e^{U_{xx}}\left[\sin(U_{xx}) + \cos(U_{xx})\right] U_{xx}^{[1]} - 2\alpha\left(UU^{[1]} + U_x U_x^{[1]}\right)\right\}\Big|_{p=0}, \tag{19}$$

$$\cdots$$

The solution $V_0^{[m]}(x)$ of above linear equation can be easily obtained:

$$V_0^{[m]}(x) = C_m \omega_\gamma(x, 0) - D_m \omega_\gamma(x, 2\pi) + \int_0^{2\pi} \omega_\gamma(x, x') f_m(x') d\, x', \tag{20}$$

where,

$$\omega_1(x, x') = -\frac{1}{2\beta} e^{-\beta|x - x'|}, \tag{21}$$

$$\omega_2(x, x') = -\frac{1}{2\beta} \sin(\beta|x - x'|), \tag{22}$$

are the fundamental solutions of MODE 1 and MODE 2, respectively. The two coefficients C_m and D_m are determined by the two boundary conditions $V_0^{[m]}(0) = V_0^{[m]}(2\pi) = 0$.

For the sake of numerical domain integral, we divide $[0, 2\pi]$ into N equal subdomains (N=500). We simply select $\beta^2 = 0.1$ and use $U_0(x) = 0$ for all values of α. Certainly, the solution of equation (14) is a function of α, shown as Figure 1, from which we can see clearly the "continuous deformation" of the solution with respect to α. The corresponding iterative

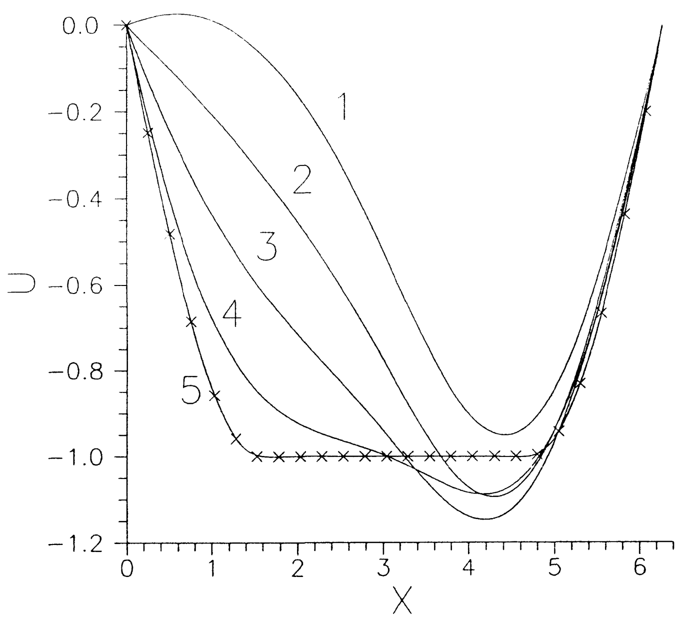

Figure 1: The solutions of equation (14) for $0 \leq \alpha \leq 1000$

Curve 1 : $\alpha = 0.0$
Curve 2 : $\alpha = 0.25$
Curve 3 : $\alpha = 1.0$
Curve 4 : $\alpha = 5.0$
Curve 5 : $\alpha = 1000$
centered symbols : $\mathcal{S}(x)$

times and the values of λ used for the different α are given in Table 1. It should be emphasized that, although the non-linear equation (14) does NOT contain any linear operators at all, we still obtain the convergent numerical results by using respectively the two different linear operators $\mathcal{L}_1$ and $\mathcal{L}_2$! Note also that the solution in case $\alpha = 1000$ is very close to the real function

$$\mathcal{S}(x) = \begin{cases} -sin(x) & \text{when } x \in [0, \pi/2], \\ -1 & \text{when } x \in [\pi/2, 3\pi/2], \\ -sin(2\pi - x) & \text{when } x \in [3\pi/2, 2\pi], \end{cases} \tag{23}$$

shown as Figure 1. This result is reasonable, because $\mathcal{S}(x)$ is certainly a solution of equation (14) in case α tends to infinity. Therefore, we have many reasons to believe that what we have obtained are indeed the solutions of equation (14) for $0 \leq \alpha \leq 1000$.

We can simply compare the proposed BEM with that given in [5].

Similarly, a one-dimentional non-linear equation $U_{xx} = \alpha U^2$ ($x \in [0,1]$) with the two boundary conditions $U(0) = 1, U(1) = 0.25$ has been used in [5] as one simple example: the domain is firstly divided into N subdomains and then the non-linear equation is *linearized* in each subdomain. As a result, a set of 2N algebraic equations is needed to be solved which needs certainly much CPU, if N is large, for instance, N = 500, as used in this paper. But, the proposed BEM needs only solve a set of TWO algebraic equations about the two unknown coefficients at each iteration. Therefore, the proposed BEM seems much more efficient than that given in [5].

Our computational tool is COMPAQ Prolinra 4/50 and double precision variables are used. In this paper, we use two kinds of convergence criterion

$$RMS_1 = \sqrt{\frac{\sum_{i=0}^{N} \{\mathcal{A}[U(x_i)] - f(x_i)\}^2}{N+1}} < 10^{-3} \tag{24}$$

and

$$RMS_2 = \sqrt{\frac{\sum_{i=0}^{N} |\delta U(x_i)|^2}{N+1}} < 10^{-5}. \tag{25}$$

Iteration will be stoped if either of them is satisfied.

4 Conclusions and Discussions

In this paper, the basic ideas described in [4] are greatly generalized to give a kind of new Boundary Element Method (BEM) for quite general non-linear differential operators which may NOT contain any linear operators at all. This kind of new BEM has the following advantages:

(A) it can be used to solve those non-linear problems which do NOT contain any linear operators at all;

(B) in any cases, we have great freedom to select a sort of proper and simple linear operator $\mathcal{L}$ whose fundamental solution is familiar to us, especially when $\mathcal{L}_0$ does not exist, or when $\mathcal{L}_0$ is so complex that its corresponding fundamental solution is either unknown or difficult to be obtained;

(C) the traditional BEM is only a special case of the proposed BEM so that there exists a kind of logical continuation between the traditional and the proposed BEM. This kind of logical continuation has been proved again and again to be very important in many fields of mathematics.

One simple example is applied to illustrate the effectiveness of the proposed method, which indicates that the linear operator $\mathcal{L}_0$, which is corresponding to the linear terms of the considered nonlinear problem and is very important for the traditional BEM, has now no special meaning at all — it is nothing but only a very common one of many proper linear operators suited to the proposed BEM. It illustrates also that the proposed BEM can give very good numerical approximations of a nonlinear problem which does NOT contain any linear operators at all, even in case that the nonlinearity is very strong! Note that we have now great freedom to select a proper and simple linear operator, whose fundamental solution is familiar to us, for the proposed BEM. Certainly, the greater freedom means the better, this is exactly the reason why the proposed BEM seems to be superior to the traditional one. However, how can we use this kind of freedom ? That is to say, how can we know a selected linear operator is *proper* and is *better* than another one ? Certainly, for any a non-linear problem, there should exist many proper linear operators suited to the proposed BEM, all of which would construct a mathematical space **S**. It seems that there should exist the best linear operator in the space **S**. But how to find out the best one ? Unfortunately, we know now nearly nothing about these interesting questions. Therefore, some deep mathematical researches are necessary. On the other hand, although this example has indeed illustrated the effectiveness of the proposed BEM, it seems to be too simple (a more complex example about 2D viscous flow has been given in [4]). So, the proposed BEM must be applied to solve more complex 2D and 3D non-linear problems in engineering so as to examine and improve it.

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