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Resolution of the pressure equation by the boundary element method in multistaged turbomachineries

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ABSTRACT

The application of the BEM to the resolution of the pressure equation in a multistaged turbomachinery is presented. The numerical treatment of the pressure equation is briefly described. Some improvements to the classical BEM are presented that ensures the accuracy of the results. They are necessary to ensure a good accuracy of the results in a curved geometry as encountered in turbomachinery. The technique of resolution in a complete stage of turbomachinery (stator+rotor) is presented. The aim is to solve the pressure equation in such a case as easily as in the case of a simple row. Firstly a subdomain technique is presented that allows to take into account a moving mesh. Secondly the management of the rotation of the rotor is described and a result on a simple geometry is presented.

NOTATIONS

P	pressure
p	correction of pressure
u	flow velocity
ρ	volumic mass
r	distance between the impact point and a point of the surface element
u*	fundamental function $(\frac{1}{4\pi r})$
z	smallest distance between the impact point and an element
l_1, l_2	lengths of a surface element
Ng	number of quadrature points
ϖ	solid angle

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E' error made on the estimation of the solid angle
G matrix of the Neuman coeffcients
H matrix of the Dirichlet coefficients
i index of points on the boundary
l index of elements on the boundary
k index of points on a surface element

INTRODUCTION

The Boundary Element Method has been proven to be a powerful tool for analyzing problems of engineering and scientific concern. The range of BEM applications for solving industrial problems has grown in recent years. We will interest here in a particular field of application that concerns the resolution of the pressure equation in multistaged turbomachinery.

The purpose of this study is not to discuss the advantages or drawbacks of the BEM applied to the resolution of the pressure equation. The first aim of this paper is to show simple and efficient tools that may be implemented in a BEM program to improve its accuracy, diminish the CPU time and to describe curved geometries. The second aim is to show the treatment of a moving mesh with the BEM.

The geometries encountered in turbomachineries are very complex. Firstly, the curvature of blades, hub and shroud may be very important. This implies to use discretisation elements that may describe curved geometries very accurately. Secondly, the description of the flow in the unbladed region (between rotor and stator) does not need a very refined mesh. To ensure the accuracy of the method in such cases, some numerical improvements have been performed. The precision of the integrals appearing in the BEM and the description of the geometry with C1 will be studied.

The treatment of a multistaged turbomachinery implies the use of a moving mesh that will take into account the rotation of the rotors. We will show a technique which allows to treat a complete stage (stator+rotor) without calculating the influence matrices at each displacement of the rotor.

FORMULATION

The verification of the continuity equation may be difficult, at low Mach number or for incompressible flows, in a Navier-Stokes solver, because of its mathematical nature. The pressure correction method is a way of solving this problem (See Hirsch [1]) to reduce the error residual in the continuity equation. The principle of the method is briefly summarized below for a non-viscous flow. The pressure equation (1) is deduced classically from the momentum conservation equation:

$$\rho\frac{\partial u}{\partial t}+\rho u.\nabla u=-\nabla P \tag{1}$$

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The divergence of this equation produces a Laplacian operator for the pressure:

$$\Delta \; P \; = \; -div\left(\rho\frac{\partial \; u}{\partial \; t} + \rho \, u . \nabla u\right) \tag{2}$$

In the right hand side of this equation, it is possible to put together the terms depending on the quantity $\Theta = div(\rho V)$. This leads to a new pressure equation:

$$\Delta \; P \; = f(\Theta) + g(\rho, V) \tag{3}$$

When the continuity equation is not verified, we may introduce the residual error E:

$$E = \frac{\partial \rho}{\partial t} + \Theta \tag{4}$$

A pressure correction p is then introduced by the equation:

$$\Delta p = f(E) \tag{5}$$

The resolution of the equations is obtained with an iterative procedure. A first step consists in solving the equation of conservation of momentum and energy. A second step consists in solving the pressure correction equation. At convergence, when the pressure correction vanishes, the residual error on the continuity equation vanishes simultaneously and this equation is implicitly solved. This technique has been used by Parkinson, Pommel, Leboeuf, Ferrand [2] and Marjani [3] for parabolised Navier-Stokes equations. We will now interest in the pressure equation resolution that uses a Boundary Element Method.

The technique of resolution is based on a classical BEM as developed by Brebbia [4]. We will present here the different steps of the method very briefly. The pressure equation is written in a finite volume Ω of boundary Γ (see figure 1). The boundary conditions are of two types:

$p(M) = p_{ref}(M) \quad M \in \Gamma_1$

$\frac{\partial \; p}{\partial \; n}(M) = \left.\frac{\partial \; p}{\partial \; n}\right|_{ref}(M) \quad M \in \Gamma_2$

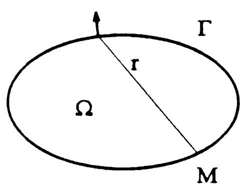

Figure 1. A classical volume of calculation.

The method consists in the integration of Poisson's equation multiplied by a fundamental function u* associated to each point M of the boundary. The Laplacian of this function has the particularity to be equal to a Dirac function at the considered point M:

$$\Delta\ u^* = \delta\ (M), \text{ where } u^* = \frac{1}{4\pi\ r} \tag{6}$$

The pressure equation, multiplied by the fundamental function, is integrated over the volume Ω. The Green theorem allows to develop this integral to obtain the following expression:

$$\int_{\Omega} \Delta\ u^* p d\omega = \int_{\Gamma} p\ \frac{\partial\ u^*}{\partial\ n} d\gamma - \int_{\Gamma} u^*\ \frac{\partial\ p}{\partial\ n}\ d\gamma + \int_{\Omega} f(E)\ u^* d\omega \tag{7}$$

Two surface equations appear involving the pressure and its gradients normal to the boundary. The influence of the right hand side of the original pressure equation appears in the volumic integral. The left hand side of this equation may be easily expressed taking into account the property of the function u*. This term gives:

$$\int_{\Omega} \Delta\ u^* p\ d\omega = \lambda\, p_M \tag{8}$$

λ is a coefficient depending on the smoothness of the geometry at the considered point. If the surface is smooth, the coefficient is 2π, if it is not, the coefficient depends on the angle formed by the tangential planes at this point. The second step of the method is the dicretization of the boundaries in some elements with the introduction of a mesh. On each element of the mesh, the pressure and its gradients are projected on a base of interpolation functions ϕ. With this method, a discrete linear equation is obtained:

$$\lambda\, p_M = H_{Mj} p_j - G_{Mlk} \left.\frac{\partial p}{\partial n}\right|_{lk} \tag{9}$$

$$H_{Mj} = \left(\int_{\gamma l} \frac{\partial\ u_M^*}{\partial\ n}\ \phi_{lk}\ d\gamma \right)_{lk=j}, G_{Mlk} = \left(\int_{\gamma l} \phi_{lk}\ u_M^* d\gamma \right)$$

$l \in \{1,..,\text{number of elements}\}, k \in \{1,..,\text{number of function per element}\}$

$j \in \{1,..,\text{number of points on the mesh}\}$

For each point of the mesh, such an equation is written, leading to the form:

$$[\lambda p] = H[p] + G\left[\frac{\partial p}{\partial n}\right] + [\textit{volumic terms}] \tag{10}$$

The coefficients of the matrices G and H represent the mutual influence of each point of the boundary. The boundary conditions are introduced in the equation giving a linear system:

$$A\begin{bmatrix} p \\ \dfrac{\partial p}{\partial n} \end{bmatrix} = B \tag{11}$$

The right hand side of the linear equation contains the volumic terms (related to the residual error E on the continuity equation) and the boundary conditions. The matrix A contains influence coefficients of matrices G and H. The resolution of the system gives the solution on the boundary. The resolution of Poisson problem with BEM needs two steps. The first one consists in the establishment of the linear system. This step needs informations concerning the geometry only and may be called the geometrical step of BEM. The second one consists in the resolution of the system and concerns aerodynamical values. We will be interested now in the geometrical step of BEM.

Firstly, we will present the numerical improvements that have been introduced in the BEM program to ensure a good accuracy of calculation. The treatment of a multistaged turbomachinery will be shown later.

THE NUMERICAL IMPROVEMENTS

The numerical techniques will be briefly presented in this paper. All the details may be found in Larrey [5].

Numerical integration

The numerical method used to calculate the influence coefficients is a classical Gauss quadrature method as described by Stroud [6]. The principle of this method leads to express the integral as a linear combination of values of the integrated function. It means:

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)\, dx\, dy = a_i\, a_j\; f(\xi_i, \xi_j),\ (i,j) \in [1,Ng]^2 \tag{12}$$

ξ_i is the coordinate of the quadrature point and a_i is its associated weight. Ng represents the number of quadrature points. This efficient method is easy to implement and to use. The accuracy of this method has been studied by many authors. Stroud [6] gives analytic expression of the absolute error depending on the number of Gauss points Ng and on the successive derivatives of the functions. In the special case of BEM, the integrated function presents the following form:

$$\int\int w(x,y)f(x,y)dxdy \tag{13}$$

where f is a continuous function and w is a kernel of the form $1/r$ or $1/r^2$. For great values of r (that means for an impact point far from the surface element) a small number of quadrature points will be necessary to obtain a good accuracy. When r becomes small without vanishing (we except the singular integrals for the moment), the level of error may be very important if no peculiar treatment is used. That is called near singular integral. It occurs when refined mesh in one direction only in the surface is used (the aspect ratio may increase) or when great elements are close to more refined region in a mesh. The Dirichlet influence coefficients will be more sensible because of the nature of the kernel $1/r^2$. Some approaches have been proposed to solve this problem. Cox and Shugar [7] propose to use repeated subdivisions of the element, Hayami and Brebbia [8] and Lutz [9] propose a transformation of the integrals based on a distance formulation (distance between point and element). We have proposed a technique that allows the automatic calculation of the quadrature points number required for a desired accuracy.

Consider a simple case of a rectangular surface element (of dimension l_1, l_2) and an impact point M at the distance z from the center of the element (figure 2). The more critical case will occur when the impact point is close to the surface element (it appears especially in the corner of the mesh) or when the distance z is small compared to the length of the element (it appears with great elements). Clearly the relative error made in the numerical calculation is a function of the number of quadrature points Ng, of the length of the element (l_1,l_2) and of the distance z between M and the surface element, that are the geometrical parameters.

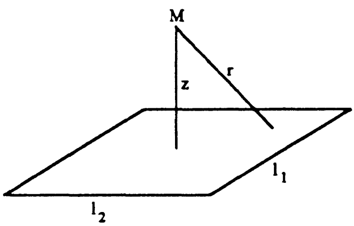

Figure 2. A simple test case

In a second step, it is necessary to define a parameter representative of all the influence coefficients and that may be calculated numerically and analytically to establish a criterion of error. To separate the problem of the numerical integration and the problem of the choice of the interpolation functions, the parameter must be independent of the interpolation functions. Consider for instance a set of interpolation functions (ϕ) whose sum is equal to 1. The sum of the Dirichlet coefficients gives:

$$\oint\!\!\oint \phi_i \frac{\vec{n}.\vec{r}}{r^3}\, d\gamma = \oint\!\!\oint \frac{\vec{n}.\vec{r}}{r^3}\, d\gamma = \varpi \tag{14}$$

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where ϖ is the solid angle defined by the point M and the surface element. It is then possible to define a relation:

$$E'(l_1,l_2,z,Ng) = \left[\frac{\varpi_n - \varpi_a}{\varpi_a}\right] \tag{15}$$

Where ϖ_n and ϖ_a are the numerical and analytical values. The choice of the interpolation functions (ϕ) will change the distribution of the error on the element but not the level. This point will be studied in the next chapter. A parametrical study has been performed to determine the expression of the relative error E'. A great number of calculations with different values of l_1 (l_2 is constant for instance and smaller as l_1), z and Ng allow to establish a map of E' depending on the different coefficients. The results may be exactly expressed with a simple relation:

$$Ng = (1 - \ln E')l_1/z \tag{16}$$

The number of quadrature points necessary to obtain a desired accuracy is proportional to the ratio l_1/z. This relation is very simple and has been successfully implemented in the program of calculation. For an impact point, the number of quadrature points may be automatically calculated for each surface element using the lengths of the element and the smallest distance between the point and the element. An other improvement has been made with the definition of two numbers of quadrature points, each of them computed with one length of the rectangular element. This leads to the expression:

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)\,dx\,dy = a_i\, b_j\, f(\xi_i, \eta_j),\ (i,j) \in [1,Ng_1]\times[1,Ng_2] \tag{17}$$

This modification will be peculiarly useful for surface elements with great aspect ratio. We will now present the results obtained with this automatic correction of quadrature points number.

The discretisation elements

Quadratic and linear elements The previous integration criterion allows a global correction of the accuracy. For the matrix H, the sum of the coefficients of a line, connected to the solid angle, may be correct. However, there is no information related to the distribution of the error on the different terms of a line of the matrix. This repartition is imposed by the choice of the discretisation surface elements.

We have shown that the interpolation of the pressure (and of its gradients) with quadratic interpolation functions produces some numerical oscillations in the resulting pressure field. These oscillations disappear when linear elements are used. But the linear elements are not able to define precisely curved geometries.

The first conclusion is that it is necessary to interpolate the pressure with linear functions. The second conclusion is that it is better to interpolate the geometry with interpolation functions of higher order.

The first idea is to use quadratic interpolation functions for the description of the geometry and linear interpolation function for the pressure. This solution is not satisfactory. It may be seen through the following example. We may try to describe the geometry of a circular arc (1/6 of circle) using Cartesian coordinates and three quadratic elements (figure 3). The result shows an error of 10% on the position of the interpolated points. The arc described by the quadratic element is a parabolic arc.

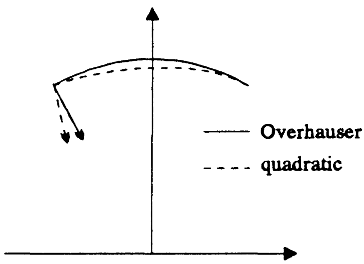

Figure 3. Description of a circular arc.

For curved geometries, it is important to keep the shape and the curvature especially at the junction of two elements. That means, for example, the continuity of the normal at the surface. If this continuity is not ensured, two normal gradients to the surface may be found on each point even for a smooth surface. The use of C^1 elements avoids this problem.

The Overhauser [10] elements have been used in graphics by Brewer and Anderson [11] and in BEM by Hall and Hibbs [12]. They ensure the continuity of the first derivative at the junction of two elements. For a mesh as represented on figure 4, the geometry inside a four points element is interpolated using the sixteen points surrounding this element.

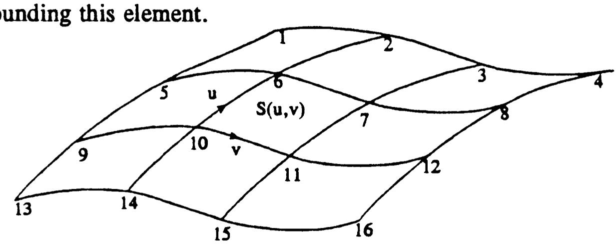

Figure 4. Overhauser Element

The surface S(u,v) defined by the four points (6,7,10,11) may be expressed using the parameters u and v. These elements allow to improve the description of the geometry without increasing the number of points of the mesh as it will be shown with the following examples. The first test consists in the calculation of the influence of a cylinder on its center. As it has been done previously, the solid angle from that the center of the cylinder sees the boundary is used as error criterion. For each section of the cylinder, 6 or 12 points are used to describe the circle (see figure 5). The calculation is made with cartesian coordinates without correction of Gauss points number (6×6 points imposed for each element). The table 1 shows the results.

Type of discretisation	error (%)
6 linear elements (6 points)	8
3 quadratic elements (6 points)	1.6
6 Overhauser elements (6 points)	0.8
6 quadratic elements (12 points)	0.1

Table 1. Error on the estimation of a solid angle associated to a cylinder and its center.

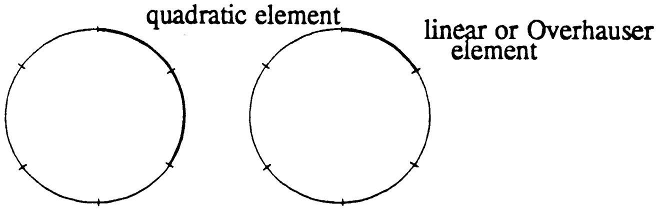

Figure 5. Discretisation of a section of a cylinder defined by 6 points.

The accuracy is increased by a ratio 10 when Overhauser elements are used instead of linear elements. To obtain better results with quadratic elements, it is necessary to increase the number of discretisation points, that is not useful in BEM. The most important point is that the correction of quadrature points may decrease the error for the Overhauser elements in a better way than for the quadratic elements because of the good description of the geometry. This may be shown by the description of a circular arc with Overhauser elements compared to the shape obtained with quadratic elements. The shape obtained with Overhauser elements and 6 points on the circle shows an error of 10^{-2} on the position of the interpolated points. The shape is very close to the circle and the normal are quite good (see figure 4).

The description of curved geometry is essential if we want to use great elements. The quadratic elements introduce an important error on the description of a curvature. The problem will be peculiarly crucial in the case of a change of curvature. This intrinsic error may not be eliminated with the Gauss points number correction. On the other side, Overhauser elements may treat this type of configuration without any difficulty.

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In conclusion, it is interesting to keep a linear interpolation of the pressure and to use C^1 interpolation for the geometry. The Overhauser elements have been chosen for their simplicity and accuracy. They have been implemented in the calculation program with success. The combination with the procedure of Gauss points number correction allows to use great curved elements without decreasing the accuracy of the calculation. It will be peculiarly interessant and necessary for the description of the curved geometries encountered in the turbomachineries. The application of the program to a multistaged turbomachinery will be now presented.

THE TREATMENT OF MULTISTAGED TURBOMACHINERIES

The resolution of the pressure equation has been extended to the case of the multistaged turbomachineries. The main problem consists in the treatment of the moving mesh necessary to describe the displacement of the rotor, the stator staying fix. We will here interest us to the case of two rows using the same blade numbers.

The configuration

For such a machine, the volume of calculation consits in two channels, one for the first row (considered as the stator), one for the second row (considered as the rotor) and a zone of periodicity (figure 6). The equation of pressure is written as follows:

$$\int_{\Omega} \frac{\Delta p}{r} = \int_{\Omega} \frac{f(E)}{r} \quad ; \; \Omega = row1 + row2 + periodicity\ \ zone \tag{18}$$

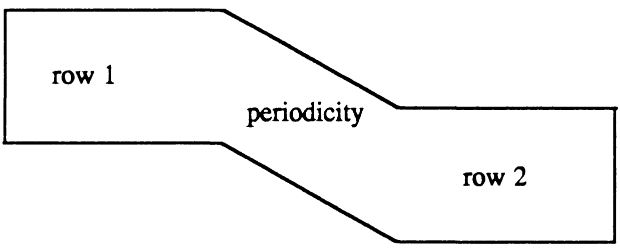

Figure 6. Volume of calculation for two rows.

When the rotor is moving, the volume calculation is changing. This implies to calculate the influence coefficients for each position. The time of calculation and the memory becomes too important. An other approach has been used to solve this problem.

The junction plane

Instead of considering the pressure equation over the whole volume, we treat the problem on two indeformable volumes such as represented on figure 7.

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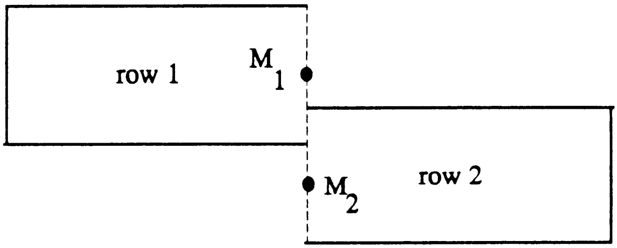

Figure 7.Introduction of the junction plane.

The equation (18) becomes:

$$\int_{row1} \frac{\Delta p}{r} = \int_{row1} \frac{f(E)}{r} \; ; \; \int_{row2} \frac{\Delta p}{r} = \int_{row2} \frac{f(E)}{r} \tag{19}$$

This new equation introduces new unknowns corresponding to the points on the bold line of the figure 7. New equations may be introduced to close the system of resolution. They appear naturally if we consider the continuity of the pressure and its gradients through the junction plane, and the periodicity of the flow. In fact, considering the periodicity of the flow, the points M_1 and M_2 are identical. For a point located on this plane we may write:

$$p_{row1} = p_{row2} ; \left.\frac{\partial p}{\partial n}\right|_{row1} = -\left.\frac{\partial p}{\partial n}\right|_{row2} \tag{20}$$

This technique allow to calculate the influence matrices at he beginning of the resolution and to solve the pressure equation for each position of the rotor without recalculating the coefficients. The last problem consists in the treatment of the rotor displacement.

The rotor displacement

The resolution of the pressure equation will be done for the two rows on the same time. It means that a unic system involving all the unknowns has to be written. The continuity equations may be introduced carefully to permit an automatic writting of the matrice for each position of the rotor.

For each row, a linear system is obtained. The points of each row are divided into two sets. The first one contains the points of the surface that are not on the plane represented with dashed line on figure 7. The second one contains the points located on this plane. The resulting matrix will be written as follows:

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DIAG	VERTI
HORIZ	DIAG2

Figure 8. Matrix for each row.

The matrix DIAG contains the influence coefficients related to the points of the first set only, that means points that are not on the junction plane. The matrices HORIZ and VERTI contain the coefficients concerning the influence between points of the first set and points of the second set. DIAG2 contains influence coefficients related to the points of the junction plane only.

The matrix of the complete system will be written in a peculiar form defined as follows:

<table>
<tr><td>Diag</td><td>0</td><td rowspan="2">A</td></tr>
<tr><td>0</td><td>Diag</td></tr>
<tr><td>Horiz</td><td>0</td><td rowspan="2">B</td></tr>
<tr><td>0</td><td>Horiz</td></tr>
</table>

Figure 9. The matrix of the complete problem.

The matrices A and B contain coefficients of the matrices Verti and Diag2 combined following the rules of the continuity relations. A table containing the correspondance relations between the points of the row 1 and those of the row 2 is written at each displacement of the rotor, and the matrices A and B are formed. In conclusion the displacement of the rotor is took into account by a change of the columns of matrices A and B. All the details may be found in Larrey [5].

This technique is very efficient for many reasons. Firstly, the management of the table of correspondance is very easy. Secondly, the structure of the final matrix allow an easy stockage. It is important to notice that the size of the final system is smaller than it was for a calculation volume represented on figure 6. If each rows contains N points, for a complete volume (figure 6) the order of the system is N^2. For the two sub-domains, the order of the final system is $2N^2$. The size of the problem is divided by the number of rows. This represent a great advantage, espacially for machines containing a wide number of rows. It is now necessary to prove that the introduction of subdomains did not decrease the accuracy of the calculation.

A simple test

No result concerning the resolution of the Navier-Stokes equations in a multistaged turbomachinery is now available. The BEM program has just been tested in a potential configuration (or with an arbitrary volumic source term). These tests have proved the accuracy of the BEM, not the capability of the Navier-Stokes solver. The equation solved by the BEM program is

$$\Delta\phi = 0 \tag{21}$$

We consider two rows, with plane radial blades as represented on figure 10. The mesh contains 200 points for each row.

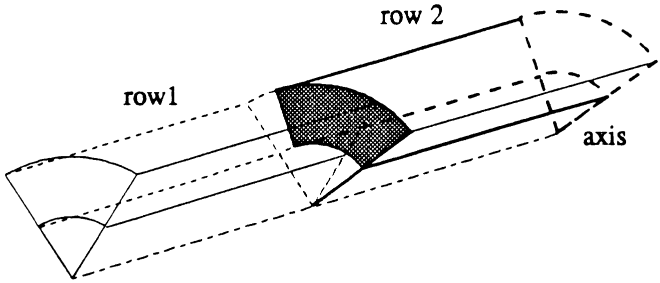

Figure 10. The geometry of the two channels.

The boundary conditions are the following:

for a blade: $\dfrac{\partial \phi}{\partial n} = 0$

for a periodicty zone: $\phi_1 = \phi_2 \,;\; \left.\dfrac{\partial \phi}{\partial n}\right|_1 = -\left.\dfrac{\partial \phi}{\partial n}\right|_2$

at inlet and outlet: $\phi_{inlet} = 1, \phi_{outlet} = 3$

The result is represented on figure 11. it shows that no discontinuity is observed through the junction plane (an accuracy of 10^{-6} is observered on each point).

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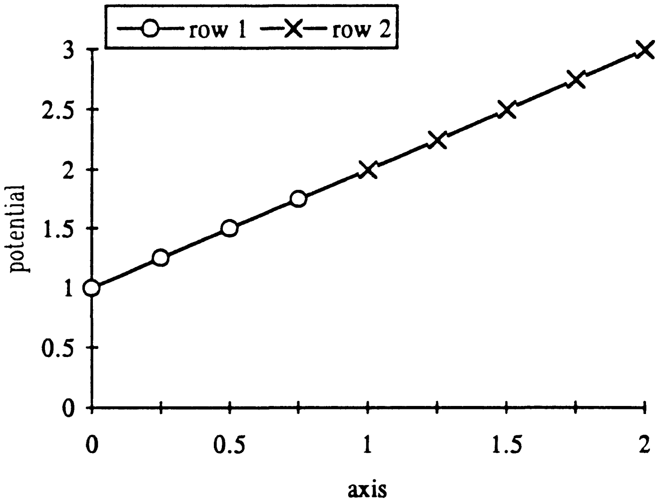

Figure 11. Result of the potential calculation on two rows.

Some other tests have been performed changing the boundary conditions with the same level of accuracy. These results show that the introduction of subdomains do not decrease the accuracy

CONCLUDING REMARKS

Some numerical improvements of the BEM have been presented. They allow to use great curved elements with a good accuracy. These techniques are necessary to represent hub and shroud of turbomachinery. They allow to decrease the number of points in the unbladed zones were the pressure on the boundary is unnecessary.

An efficient BEM program has been developped to solve the three-dimensionnal pressure equation in a multi-staged turbomachinery. A technique of subdomain allows to define a fixed mesh for each row. It introduces a junction plane where periodicity conditions and continuity relations are used to close the system. The displacement of the rotor is took into account through the displacement of a part of the columns of the final system. It is also easy to treat a moving mesh

The results of the tests show that the introduction of subdomains does not introduce a decrease of the precision. This technique allow to divide the memory size by the number of row. The BEM program is now ready to be used in a Navier-Stokes solver.

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