NUMERICAL SCHEMES FOR VORTEX SHEET INTENSITY COMPUTATION IN VORTEX METHODS TAKING INTO ACCOUNT THE CURVILINEARITY OF THE AIRFOIL SURFACE LINE

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ABSTRACT

In vortex methods, vorticity is the primary computed variable. The problem of the accuracy improvement of vorticity generation simulation at the airfoil surface line in 2D vortex methods is considered. The generated vorticity is simulated by a thin vortex sheet at the airfoil surface line, and it is necessary to determine the intensity of this sheet at each time step. It can be found from the no-slip boundary condition, which leads to a vector boundary integral equation. There are two approaches to satisfy this equation: the first one leads to a singular integral equation of the 1st kind, while the second one leads to a Fredholm-type integral equation of the 2nd kind with bounded kernel for smooth airfoils. Usually, for numerical solution of the boundary integral equation, the airfoil surface line is replaced by a polygon, which consists of straight segments (panels). A discrete analogue of the integral equation can be obtained using the Galerkin method. Different families of basis and projection functions lead to numerical schemes with different complexity and accuracy. For example, a numerical scheme with piecewise-constant basis functions provides the first order of accuracy for vortex sheet intensity, and a numerical scheme with piecewise-linear functions gives the second order of accuracy. However, the velocity field near the airfoil surface line is also of interest. In the case of rectilinear airfoil surface line discretization, the accuracy of velocity field reconstruction has no more than the first order of accuracy for both, piecewise-constant and piecewise-linear numerical schemes. In order to obtain a higher order of accuracy for velocity field reconstruction, it is necessary to take into account the curvilinearity of the airfoil surface line. In this research, we have developed such an approach, which provides the second order of accuracy both, for vortex sheet intensity computation and velocity field reconstruction.
Keywords: vortex method, boundary integral equation, vortex sheet, curvilinear panel.

1 INTRODUCTION

There are many different methods for the numerical simulation of the interaction between structures and flow, the most popular are mesh methods. The most common of them are finite difference method, finite element method, finite volume method. Within the framework of such approaches, there are many modifications that allow one to consider models of diverse complexity, taking into account different physical effects. However, often in engineering applications there is a need to analyze a large number of possible design parameters. In this case, it is impossible to perform detailed investigation for all parameter values, so there is a need to involve approximate, but cost efficient methods for the primary rough analysis of the parameters. Meshless Lagrangian vortex methods [1]–[6] fall into the category of such approaches. Their range of applicability is limited by the model of incompressible flow, however, in this field, they can be much more efficient in comparison with mesh methods [7].

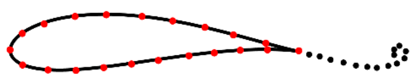
Vortex methods are based on the consideration of vorticity as the primary calculated value. There is no new vorticity generation in the flow region; new vorticity is generated only at the airfoil surface line: the generated vorticity is simulated by a vortex sheet. This vorticity then becomes part of the vortex wake as a separate vortex elements, which move in the flow

WIT Transactions on Engineering Sciences, Vol 126, © 2019 WIT Press
www.witpress.com, ISSN 1743-3533 (on-line)
doi:10.2495/BE420241

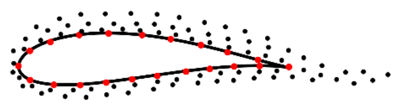
with some velocity. The intensity of this vortex sheet can be determined from the boundary condition at the airfoil surface line [8], which is expressed by the boundary integral equation.

Numerical schemes which are commonly used in vortex methods for approximate solution of boundary integral equation, imply the splitting the airfoil surface line into segments (usually called "panels"). Then, a system of linear algebraic equations approximating the integral equation is constructed. The accuracy of the simulation of the vorticity generation process directly affects the accuracy of the whole problem solution.

In the case of using the of well-known and well-studied method of discrete vortices (MDV) [2], which is applicable only for inviscid flow simulation, vorticity is being shed from the airfoil surface to the vortex wake only at some preliminary known points of separation, which are often taken at sharp edges or angle points (Fig. 1(a)). However, modern and more complicated modifications of vortex methods, such as the method of viscous vortex domains (VVD) [9], which allows one to simulate viscous flows, imply the vorticity shedding from the whole airfoil surface line (Fig. 1(b)). So for such modification, in addition to the accuracy of the vortex sheet intensity calculation, the accuracy of velocity field reconstruction in the neighborhood of the airfoil surface line is also of interest since it determines the accuracy of vorticity evolution simulation in the flow.



(a)



(b)

Figure 1: Vortices separation in (a) MDV methods; and (b) VVD methods.

In the present research, the method of Viscous Vortex Domains is considered as a basic modification of vortex methods. In the framework of VVD, vorticity in the flow domain is represented by a set of vortex elements having constant (in time) circulation and moving with velocity $\vec{V}+\vec{W}$, where $\vec{V}$ is the flow velocity and $\vec{W}$ is the so-called "diffusive velocity" [9].

Normally, in vortex methods an airfoil surface line is approximated by rectilinear panels [10]. For this case, numerical schemes for a boundary integral equation approximate solution of different accuracy and the corresponding numerical complexity can be constructed. For example, in [11] several schemes based on the Galerkin method ideas are developed. In these schemes, the vortex sheet intensity is represented by a piecewise-constant or piecewise-linear function. Note that in [12] a piecewise-linear scheme was also constructed, however, for other reasons. In [11] it is shown that piecewise-constant and piecewise-linear schemes provide the 1st and the 2nd order of accuracy for the vortex sheet intensity, respectively. However, the piecewise-linear scheme does not permit one to provide the 2nd order of accuracy for the velocity field reconstruction in neighborhood of the airfoil surface line. This is due to the fact that, in the case of the airfoil surface line approximation with rectilinear panels instead of the original smooth curve, we deal with dummy angular points between the panels. Their presence effects significantly the velocity field generated by the vortex sheet which is located on such rectilinear panels. It leads to the loss of smoothness of the velocity field in proximity to the angle points, so only 1st order of accuracy can be achieved. Therefore, it seems that it is necessary to take into account the curvilinearity of the airfoil surface line to improve the order of accuracy of the velocity field reconstruction.

The aim of this research is the accuracy analysis for vortex sheet intensity computation and velocity field reconstruction when the airfoil surface line is approximated by rectilinear and curvilinear panels.

2 THE PROBLEM STATEMENT

In vortex methods, the vortex sheet intensity at the airfoil surface line K can be determined from the no-slip boundary condition. This condition, together with the generalized Helmholtz decomposition, leads to the vector boundary integral equation with respect to the vortex sheet intensity [8], [13]. This equation can be reduced to a scalar integral equation by projecting it onto a normal or a tangent vector at the airfoil surface line [8]. In the first case, we obtain a singular integral equation, in the second case – a Fredholm-type integral equation of the second kind. In [14], it is shown that the second approach allows one to obtain a solution with higher accuracy. According to this approach, the boundary integral equation with respect to the vortex sheet intensity $\gamma(r)$ is as follows [8], [11], [14]

$$\oint_K Q_\tau(\vec{r},\vec{\xi})\gamma(\vec{\xi})dl_\xi - \frac{1}{2}\gamma(\vec{r}) = f_\tau(\vec{r}), \qquad \vec{r} \in K. \tag{1}$$

Here,

$$Q_\tau(\vec{r},\vec{\xi}) = \frac{\vec{n}(\vec{r})\cdot(\vec{r}-\vec{\xi})}{2\pi\,|\,\vec{r}-\vec{\xi}\,|^2}, \tag{2}$$

$\vec{n}$ is unit outward normal vector to the airfoil surface line, and f_τ is the right-hand side which depends on the vorticity distribution in the flow region, the airfoil surface line velocity and the incident flow velocity. We assume that the right-hand side is known function, which corresponds to the case when the velocity of the airfoil surface line is known function or when the coupled hydroelastic problem is solved according to partitioned approach. In the more complex case where the coupled hydroelastic problem is solved using a monolithic approach, the airfoil velocity is also an unknown variable, and motion equation for the airfoil surface line should be solved together with the boundary integral equation.

Eqn (1) has infinite set of solutions; in order to select the unique one, the additional condition should be added

$$\oint_K \gamma(\vec{r})dl_r = \Gamma, \tag{3}$$

where Γ is given value of total circulation around the airfoil.

Note that the kernel $Q_\tau(\vec{r},\vec{\xi})$ of eqn (1) is bounded for smooth airfoils; if the surface line is C^2-smooth curve, it is easy to prove that

$$\lim_{|\vec{r}-\vec{\xi}|\to 0} |\,Q_\tau(\vec{r},\vec{\xi})\,| = \frac{\varkappa}{4\pi},$$

where $\varkappa$ is the curvature of the curve K at the corresponding point.

If there is angle poin at the airfoil surface line, the kernel is unbounded in this point, and exact solution for the vortex sheet intensity has a weak singularity [15]. If the initial airfoil surface line has an angular point, this singularity is physically justified and represents real flow specificity. However, when we discretize the airfoil surface line by rectilinear panels, we obtain a set of "dummy" angle points (Fig. 2). When the numerical solution is assumed to be piecewise-constant along the panels, or even piecewise-linear, but the neighboring panels have nearly the same lengths, airfoil approximation with the polygon is acceptable and the convergence of the numerical solution to the exact one takes place for vortex sheet intensity with the 1st and 2nd order of accuracy, respectively. For significantly different lengths of neighboring panels, the result of numerical solution can be far from exact solution, not only quantitatively, but even qualitatively.

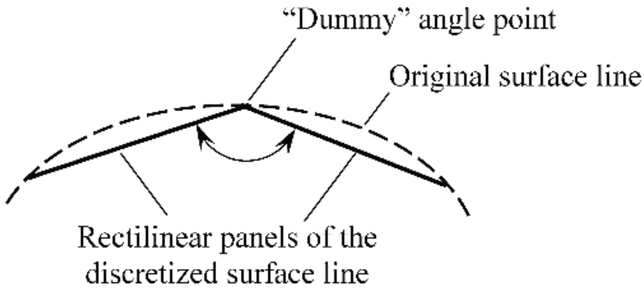


Figure 2: Dummy angle point formation between rectilinear panels.

Thus, in order to increase the accuracy vortex sheet intensity computation, it is necessary to avoid the appearance of artificial angle points, i.e. to take into account the curvilinearity of the airfoil surface line.

The flow velocity can be reconstructed using the Biot–Savart law through known vorticity distribution $\vec{\Omega}(\vec{r})$ in the flow domain S, the intensity of the vortex sheet $\vec{\gamma}(\vec{r})$ on the airfoil surface line K, and the velocity of airfoil surface line. In the considered particular case of the immovable airfoil, it takes the following form

$$\vec{V}(\vec{r}) = \vec{V}_\infty + \int_S \frac{\vec{\Omega}(\vec{r}) \times (\vec{r} - \vec{\xi})}{2\pi |\vec{r} - \vec{\xi}|^2} dS_\xi + \oint_K \frac{\vec{\gamma}(\vec{r}) \times (\vec{r} - \vec{\xi})}{2\pi |\vec{r} - \vec{\xi}|^2} dl_\xi,$$

where $\vec{V}_\infty$ is the incident flow velocity. Note that hereinafter scalar vorticity field $\Omega(\vec{r})$ and scalar vortex sheet intensity $\gamma(\vec{r})$ are considered

$$\vec{\Omega}(\vec{r}) = \Omega(\vec{r})\vec{k}, \quad \vec{\gamma}(\vec{r}) = \gamma(\vec{r})\vec{k},$$

where k is unit vector orthogonal to the flow plane.

More generally, when the airfoil surface line is movable, the flow velocity according to the generalized Helmholtz decomposition also depends on the airfoil surface line velocity [8].

3 THE NUMERICAL METHOD

For the numerical solution of the boundary integral equation, we use the basic ideas developed in [11], where the Galerkin approach is applied to numerical schemes construction. In [11], schemes with a rectilinear discretization of the airfoil surface line are described, an approximate solution is represented as piecewise-constant or piecewise-linear function. Here, we briefly describe such schemes and then generalize main ideas to the case of curvilinear panels [16].

3.1 Numerical scheme with rectilinear panels

Let the airfoil surface line be discretized by rectilinear panels K_i, $i=1,\dots,N$. Then, the system of basis functions $\varphi_i^q(\vec{r})$, $i=1,\dots,N$, $q=0,1$, is introduced such that

$$\varphi_i^0(\vec{r})=\begin{cases}1, & \vec{r}\in K_i,\\ 0, & \vec{r}\notin K_i;\end{cases}\qquad \varphi_i^1(\vec{r})=\begin{cases}\dfrac{(\vec{r}-\vec{c}_i)\cdot\vec{\tau}_i}{L_i}, & \vec{r}\in K_i,\\ 0, & \vec{r}\notin K_i,\end{cases} \tag{4}$$

where $\vec{c}_i$ is the center of the i-th panel; L_i is the length of the i-th panel.

Then, the piecewise-constant and piecewise-linear numerical solutions are represented as linear combinations of the basis functions with unknown coefficients $\{\gamma_i^0\}_{i=1}^N$ and $\{\gamma_i^1\}_{i=1}^N$

$$\gamma^c(\vec{r})=\sum_{i=1}^N\gamma_i^0\varphi_i^0(\vec{r}),\quad \gamma^l(\vec{r})=\sum_{i=1}^N(\gamma_i^0\varphi_i^0(\vec{r})+\gamma_i^1\varphi_i^1(\vec{r})),\quad \vec{r}\in K. \tag{5}$$

All the formulae below are presented for piecewise-linear solution, however, they can be easily adapted to the piecewise-constant case.

Then, according to Galerkin approach, unknown coefficients γ_i^0 and γ_i^1 can be found from the orthogonality condition of the boundary integral equation residual to the projection functions which we choose the same as the basis functions $\varphi_i^q(s)$, $i=1,\dots,N$, $q=0,1$ [11], [14]. The residual of eqn (1) on the i-th panel has the following form

$$z_i(\vec{r})=\sum_{j=1}^N\left(\gamma_j^0\int_{K_j}Q_\tau(\vec{r},\vec{\xi})\varphi_j^0(\vec{\xi})dl_\xi+\gamma_j^1\int_{K_j}Q_\tau(\vec{r},\vec{\xi})\varphi_j^1(\vec{\xi})dl_\xi\right)-\frac{1}{2}\left(\gamma_i^0\varphi_i^0(\vec{r})+\gamma_i^1\varphi_i^1(\vec{r})\right)-f_\tau(\vec{r}).$$

Its orthogonality condition to the basis functions leads to a system of linear equations

$$\int_{K_i}z_i(\vec{r})\varphi_i^p(\vec{r})dl_r=0,\quad i=1,\dots,N,\quad p=0,1,$$

which, in turn, can be written down in the following form

$$\sum_{j=1}^N\left(\gamma_j^0\int_{K_i}\varphi_i^p(\vec{r})dl_r\int_{K_j}Q_\tau(\vec{r},\vec{\xi})\varphi_j^0(\vec{\xi})dl_\xi+\gamma_j^1\int_{K_i}\varphi_i^p(\vec{r})dl_r\int_{K_j}Q_\tau(\vec{r},\vec{\xi})\varphi_j^1(\vec{\xi})dl_\xi\right)-$$
$$-\frac{1}{2}\left(\gamma_i^0\int_{K_i}\varphi_i^0(\vec{r})\varphi_i^p(\vec{r})dl_r+\gamma_i^1\int_{K_i}\varphi_i^1(\vec{r})\varphi_i^p(\vec{r})dl_r\right)=$$
$$=\int_{K_i}f_\tau(\vec{r})\varphi_i^p(\vec{r})dl_r,\quad i=1,\dots,N,\quad p=0,1. \tag{6}$$

Eqn (3) is approximated straightforwardly

$$\sum_{i=1}^N\int_{K_i}\left(\gamma_i^0\varphi_i^0(\vec{r})+\gamma_i^1\varphi_i^1(\vec{r})\right)dl_r=\Gamma. \tag{7}$$

The system which consists of linear eqns (6) and (7) is overdetermined; we regularize it similarly to [2] by adding the so-called regularization variable R to those of eqn (6) which correspond to $p=0$. In block-matrix form regularized linear system can be written down as

$$\begin{pmatrix} A^{00}+D^{00} & A^{01}+D^{01} & I \\ A^{10}+D^{10} & A^{11}+D^{11} & O \\ L^{0} & L^{1} & 0 \end{pmatrix}\begin{pmatrix} \gamma^{0} \\ \gamma^{1} \\ R \end{pmatrix}=\begin{pmatrix} b^{0} \\ b^{1} \\ \Gamma \end{pmatrix}. \tag{8}$$

Here A^{pq} are $N \times N$ square matrices; D^{pq} are diagonal matrices, $p,q=0,1$; I and O are vectors consist of ones and zeros, respectively; L^0 and L^1 are vectors consist of integrals from the corresponding basis functions along the curvilinear panels; $\gamma^0=(\gamma_1^0,\dots,\gamma_N^0)^T$ and $\gamma^1=(\gamma_1^1,\dots,\gamma_N^1)^T$ are vectors of unknown coefficients; R is regularization variable; b^0 and b^1 are right-hand side vectors

$$\begin{gathered} A_{ij}^{pq}=\int_{K_i}\left(\int_{K_j} Q_\tau(\vec{r},\vec{\xi})\varphi_j^q(\vec{\xi})dl_\xi\right)\varphi_i^p(\vec{r})dl_r, \quad D_{ii}^{pq}=-\frac{1}{2}\int_{K_i}\varphi_i^p(\vec{r})\varphi_i^q(\vec{r})dl_r, \\ L_i^p=\int_{K_i}\varphi_i^p(\vec{r})dl_r, \quad b_i^p=\int_{K_i} f_\tau(\vec{r})\varphi_i^p(\vec{r})dl_r, \qquad i,j=1,\dots,N, \quad p,q=0,1. \end{gathered} \tag{9}$$

The scheme (8), which corresponds to a piecewise-linear approximate solution, can be reduced to a scheme for a piecewise-constant solution

$$\begin{pmatrix} A^{00}+D^{00} & I \\ L^{0} & 0 \end{pmatrix}\begin{pmatrix} \gamma \\ R \end{pmatrix}=\begin{pmatrix} b^{0} \\ \Gamma \end{pmatrix}, \tag{10}$$

where the coefficients, again, have the form (9).

The coefficients (9) can be calculated analytically; all the formulae for their calculation are given in [17].

3.1 Numerical scheme with curvilinear panels

Let us suppose that the airfoil surface line is parameterized with the natural parameter s – the arc length of the curve. Then, eqn (1) takes the form

$$\int_K Q_\tau(s,\sigma)\gamma(\sigma)d\sigma-\frac{1}{2}\gamma(s)=f_\tau(s), \quad s\in[0,L].$$

Here, L is the length of the airfoil surface line.

Let the airfoil surface line now be discretized by N curvilinear panels whose endings and beginnings correspond to the values of the parameter s. Then, the parameter s at the i-th panel varies over the range $[s_{i-1},s_i]$.

The basis functions $\varphi_i^q(s)$, $i=1,\dots,N$, $q=0,1$, similar to (4), are introduced, but now they depend on the arc length

$$\varphi_i^0(s)=\begin{cases} 1, & s\in[s_{i-1},s_i], \\ 0, & s\notin[s_{i-1},s_i]; \end{cases} \qquad \varphi_i^1(s)=\begin{cases} \dfrac{s-s_i^*}{\delta_i}, & s\in[s_{i-1},s_i], \\ 0, & s\notin[s_{i-1},s_i]. \end{cases} \tag{11}$$

Here, $\delta_i=s_i-s_{i-1}$ is the length of the i-th curvilinear panel, the parameter value $s_i^*=\dfrac{s_i+s_{i-1}}{2}$ corresponds to the center of the i-th panel.

The approximate solution on the airfoil surface line has the same form as earlier (5); the orthogonality condition of the residual of the equation to the basis functions leads to the systems which coincide with (10) or (8), respectively; the coefficients have the following form

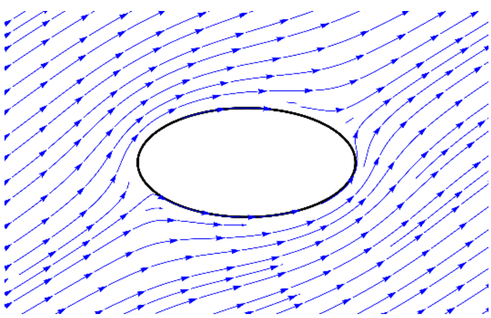
$$
\begin{aligned}
&A_{ij}^{pq} = \int_{s_{i-1}}^{s_i} \left(\int_{s_{j-1}}^{s_j} Q_\tau(s,\sigma)\varphi_j^q(\sigma)d\sigma \right) \varphi_i^p(s)ds, \quad D_{ii}^{pq} = -\frac{1}{2}\int_{s_{i-1}}^{s_i} \varphi_i^p(s)\varphi_i^q(s)ds, \\
&L_i^p = \int_{s_{i-1}}^{s_i} \varphi_i^p(s)ds, \quad b_i^p = \int_{s_{i-1}}^{s_i} f_\tau(s)\varphi_i^p(s)ds, \qquad i,j=1,\dots,N, \quad p,q=0,1.
\end{aligned}
\tag{12}
$$

As earlier, a piecewise-constant scheme can be constructed, similarly to (10).

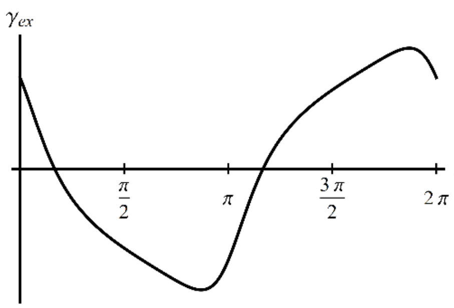
Of course, the calculation of coefficients (12), which now requires the integration over the curvilinear panels, is quite complicated problem. In contrast with case of rectilinear panels, it is not possible to obtain exact formulae for their calculation. In [18] a methodology has been developed that allows to calculate them approximately. Using the expanding of the integrands into Taylor series, all the necessary approximate formulae were obtained for integrals calculation over curvilinear panels panels.

4 NUMERICAL EXPERIMENTS

In order to compare the described schemes with rectilinear and curvilinear airfoil surface line discretization and piecewise-constant and piecewise-linear vortex sheet intensity representation, we consider the model problem of potential flow simulation around elliptical airfoil (Fig. 3(a)).



(a)



(b)

Figure 3: (a) Flow around elliptical airfoil; and (b) Exact solution for vortex sheet intensity.

Let us suppose that the surface line of elliptical airfoil with semiaxes a_1 and b_1 is given by the parametric relations

$$
\begin{cases} x(t) = a_1 \cos t, \\ y(t) = b_1 \sin t, \end{cases} \qquad t \in [0;2\pi).
$$

The incident flow has velocity $|\vec{V}_\infty|$ and angle of incidence β ; there is no vorticity in the flow region.

In this case, complex potential of the flow has the following form

$$f(z) = W_\infty^* \hat{\zeta}(z) + \frac{(a_1 + b_1)^2 W_\infty}{\hat{\zeta}(z)}, \quad z = x + iy,$$

where $W_\infty = \frac{1}{2} | V_\infty | e^{i\beta}$; $\hat{\zeta}(z) = z - \sqrt{z^2 - a^2}$; "$*$" means complex conjugate. From this complex potential, the velocity field can be calculated: $V^{ex} = f'(z)^*$.

Also, the exact solution for vortex sheet intensity for such problem can be derived

$$\gamma^{ex}(t) = \frac{| V_\infty | \sin(\beta - t)}{\sqrt{a_1^2 \sin^2 t + b_1^2 \cos^2 t}}, \qquad t \in [0; 2\pi).$$

The following values of parameters were chosen: incident flow velocity $|\vec{V}_\infty| = 1$, the angle of incidence $\beta = \pi / 6$, ellipse semiaxes $a = 1.0$ and $b = 0.5$. The exact solution for such parameters is shown in Fig. 3(b).

The airfoil surface line in numerical experiments was discretized into N panels with approximately the same lengths equal to δ.

The error of vortex sheet intensity computation is estimated using the L_1 norm

$$\| \varDelta\gamma \|_{L_1} = \oint_K | \gamma^{ex}(\vec{r}) - \gamma(\vec{r}) | dl_r.$$

In [19], the method for this integral calculation is described: it is shown there how the approximate solution at the approximated airfoil can be projected onto the original airfoil surface line.

The error of the velocity field reconstruction in neighborhood of the airfoil can be estimated in the similar way

$$\| \varDelta V \|_{L_1} = \oint_{K^h} | \vec{V}^{ex}(\vec{r}) - \vec{V}(\vec{r}) | dl_{K^h}.$$

Here, the integral is calculated over the curve K^h surrounding the boundary of the airfoil and spaced from the airfoil surface line by a value $\frac{1}{8}\delta$; $\vec{V}^{ex}(\vec{r})$ is the exact velocity field around the elliptical airfoil; $\vec{V}(\vec{r})$ is the velocity field reconstructed according to the Biot–Savart law through the numerical solution $\gamma(\vec{r})$.

The errors of vortex sheet intensity computation and velocity field reconstruction using the scheme with rectilinear panels and piecewise-constant and piecewise-linear solutions are presented in Table 1 for different number of panels. Also a'posteriori estimates for the order of accuracy are given in Table 1 as a logarithms of the errors ratio

$$m_\gamma = \log_{\frac{N_i}{N_{i-1}}} \frac{\| \Delta\gamma \|_{L_1}^{(N_{i-1})}}{\| \Delta\gamma \|_{L_1}^{(N_i)}}, \qquad m_V = \log_{\frac{N_i}{N_{i-1}}} \frac{\| \Delta V \|_{L_1}^{(N_{i-1})}}{\| \Delta V \|_{L_1}^{(N_i)}}.$$

In Table 1, the indices "c" and "l" correspond to the piecewise-constant and piecewise-linear numerical solutions, respectively.

Table 1 shows that the piecewise-constant and piecewise-linear schemes provide the first and second order of accuracy, respectively, for the vortex sheet intensity computation. It should be noted that the error of the piecewise-linear scheme is an order of magnitude lower compared to the piecewise-constant scheme.

Table 1: The errors for vortex sheet intensity computation and velocity field reconstruction using numerical schemes with rectilinear panels.

N	32	64	128	256	500	1000
$\|\| \Delta\gamma^c \|\|_{L_1}$	0.367716	0.182186	0.090671	0.045283	0.023180	0.011589
m_γ^c	–	1.01	1.00	1.00	1.00	1.00
$\|\| \Delta\gamma^l \|\|_{L_1}$	0.057839	0.017131	0.004532	0.001164	0.000309	0.000078
m_γ^l	–	1.76	1.92	1.96	1.98	1.99
$\|\| \Delta V^c \|\|_{L_1}$	0.188352	0.087041	0.042885	0.021439	0.011005	0.005460
m_V^c	–	1.11	1.02	1.00	1.00	1.01
$\|\| \Delta V^l \|\|_{L_1}$	0.137691	0.060351	0.028499	0.013966	0.007083	0.003496
m_V^l	–	1.18	1.08	1.03	1.01	1.02

However, both schemes provide only the first order of accuracy for the calculation of the velocity field near the airfoil surface line. Moreover, the use of the piecewise-linear scheme does not lead to significant decreasing of the error of velocity field calculation: the error decreases less than twice in comparison to piecewise-constant scheme.

The error values for the schemes with curvilinear panels are shown in Table 2.

Table 2: The errors for vortex sheet intensity computation and velocity field reconstruction using numerical schemes with curvilinear panels.

N	32	64	128	256	500	1000
$\|\| \Delta\gamma^c \|\|_{L_1}$	0.299910	0.150430	0.075123	0.037549	0.019225	0.009612
m_γ^c		1.00	1.00	1.00	1.00	1.00
$\|\| \Delta\gamma^l \|\|_{L_1}$	0.017236	0.004556	0.001132	0.000283	0.000074	0.000019
m_γ^l		1.92	2.01	2.00	2.00	2.00
$\|\| \Delta V^c \|\|_{L_1}$	0.149978	0.070900	0.034642	0.017184	0.008760	0.004343
m_V^c		1.08	1.03	1.01	1.01	1.01
$\|\| \Delta V^l \|\|_{L_1}$	0.011160	0.002802	0.000704	0.000176	0.000046	0.000012
m_V^l		1.99	1.99	2.00	2.00	2.01

From Table 2 it is seen that the piecewise-constant scheme, as earlier, provides the first order of accuracy both for the vortex sheet intensity and velocity field calculation. As for the piecewise-linear scheme, the usage of curvilinear panels makes it possible to improve significantly the accuracy of velocity field reconstruction: now the error has the 2nd order.

5 CONCLUSIONS

Different numerical schemes for boundary integral equation solution in vortex methods are considered. The unknown value in this integral equation is intensity of the vortex sheet at the airfoil surface line that simulates the generation of a new vorticity. The schemes differ in two factors: 1. different methods for approximate airfoil surface line description are used: by rectilinear and curvilinear panels; 2. there are two ways of numarical solution representation: in the form of piecewise-constant or piecewise-linear function.

These schemes were compared for a model problem of flow simulation around elliptical airfoil: in terms of accuracy of the vortex sheet intensity computation and in terms of the accuracy of the velocity field reconstruction near the airfoil surface line. It is shown that the piecewise-constant and piecewise-linear schemes provide the first and the second orders of accuracy, respectively, for the vortex sheet intensity both in the case of rectilinear and curvilinear discretization of the airfoil surface line. However, for the velocity field the piecewise-linear scheme with rectilinear panels does not allow one to obtain the second order of accuracy. To obtain the second order, it is necessary to take into account the curvilinearity of the airfoil surface line.

Of course, the calculation of the system coefficients in case of piecewise-linear vortex sheet intensity distribution and curvilinear panels is more complicated procedure, compared with schemes with rectilinear panels and piecewise-constant distribution. But the fact that this scheme provides results with much higher accuracy, allows one to use much fewer panels discretizing the airfoil surface line, and, therefore, decrease the matrix size. In the case when the airfoil is deformable, the decreasing of the matrix size significantly reduces the computational complexity of the algorithm, because in this case, fully filled matrix should be inversed at each time step. In addition, the nonuniform linear distribution of the vortex sheet intensity over the panel allows one to transform the vorticity from the vortex sheet to the vortex wake with higher accuracy (in comparison with discrete or piecewise-distribution of the vortex sheet intensity), since in this case several vortex elements with different intensities can be discharged from one panel in accordance with the distribution.

ACKNOWLEDGEMENT

The financial support for the present project was partly provided by the Russian Foundation for Basic Research (RFBR) (grant No. 18-31-00245).

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