The boundary element method revisited

N. A. Dumont
Civil Engineering Department,
Pontifical Catholic University of Rio de Janeiro, Brazil

Abstract

The collocation boundary element method is derived on the basis of the weighted-residuals statement. Only the static case is addressed, as it already involves all relevant conceptual issues. The present outline brings to discussion some relevant aspects and implementation issues of the method that should belong in any text book. It is shown that, if the boundary element method is consistently formulated, an inherent error term – related to arbitrary rigid-body displacements – is naturally taken into account and has no influence on the resultant matrix equation, with traction force parameters that are always in balance independently of mesh discretization. The constitutive matrices of the method – the single-layer and double-layer potential matrices **G** and **H** – present some spectral properties that are per se interesting but that also have applicability consequences. The matrix **G** is rectangular, if consistently obtained. For and adequately formulated problem, the solution of the resultant matrix equation is always possible (and unique) whether directly or approximated in terms of equivalent nodal forces. The effects of body forces, whenever transformable to boundary actions, may be expressed in terms of the boundary interpolation functions, which renders the final matrix equation more elegant and speeds up calculations in no detriment to accuracy. There is a novel proposition for the interpolation of traction forces along curved boundaries, with results that may be only slightly improved, as compared to the classical procedure, but that simplifies numerical computation and adds to the consistency of the method in terms of patch test assessments. The conceptual and numerical developments are illustrated by means of a few examples.

Keywords: boundary elements, weighted-residual methods, generalized inverses.

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1 Introduction

The conventional, collocation boundary element method (CBEM), whenever applicable, is a simple, powerful numerical analysis tool [1]. There is an uncountable number of alternatives of formulations and implementations, some of them based on sound principles [7, 8], many of them contributing to the richness of the method [1], but unfortunately some of them just misleading. The present contribution is not a review paper and, on the contrary, makes just a few references to the technical literature. In this outline, which starts with a consistent derivation of the CBEM, one pays tribute to the method by addressing some theoretical issues that are in general overlooked. One expects to add to the conceptual consolidation of the method by analyzing some relevant spectral properties, and also daring to suggest a few improvements.

2 A simple, consistent derivation of the BEM

2.1 Problem formulation

An elastic body is submitted to body forces b_i in the domain Ω and traction forces $\bar{t}_i$ on part Γ_σ of the boundary. Displacements $\bar{u}_i$ are known on the complementary part Γ_u of Γ. One is looking for an adequate approximation of the stress field that satisfies equilibrium in the domain,

$$\sigma_{ji},_j + b_i = 0 \quad \text{in } \Omega, \tag{1}$$

also satisfying the boundary equilibrium and compatibility equations,

$$\sigma_{ji}\eta_j = \bar{t}_i \quad \text{along } \Gamma_\sigma, \quad u_i = \bar{u}_i \quad \text{on } \Gamma_u, \tag{2}$$

where η_j is the outward unit normal to Γ. Indices i, j, (also k, l) may assume values 1, 2 or 3, as they refer to the coordinate directions x, y or z, respectively, for a general 3D analysis. Sum is indicated by repeated indices. Particularization to 2D analysis as well as to potential problems is straightforward.

2.2 From a variational to a consistent weighted-residuals statement

Assuming that σ_{ij} is a symmetric tensor that satisfies a priori the constitutive equation $\sigma_{ij} = C_{ijkl}u_{k,l}$, the present problem might be formulated in the frame of the strong form of the principle of stationary (minimum) total potential energy [6], for a variation δu_i of u_i, already extending the boundary integral from Γ_σ to Γ, since, according to eqn (2), $\delta u_i = 0$ on Γ_u:

$$\delta\Pi = -\int_\Omega \left(\sigma_{ji,j} + b_i\right)\delta u_i \mathrm{d}\Omega + \int_\Gamma \left(\sigma_{ji}\eta_j - \bar{t}_i\right)\delta u_i \mathrm{d}\Gamma = 0\,. \tag{3}$$

However, one rather formulates the problem in a non-variational, less restrictive framework than eqn (3), in terms of weighted residuals, resorting to a field of *fundamental solutions*, that is, stresses and displacements of the same elasticity problem, $\delta\sigma^*_{ij} = C_{ijkl}\,\delta u^*_{k,l}$, that satisfy the homogeneous part of eqn (1), but not the boundary conditions of eqn (2):

$$-\int_\Omega \left(\sigma_{ji,j} + b_i\right)\delta u^*_i \mathrm{d}\Omega + \int_\Gamma \left(\sigma_{ji}\eta_j - t_i\right)\delta u^*_i \mathrm{d}\Gamma = 0 \,. \tag{4}$$

Integrating by parts twice the first term on the left of the above equation, with successive application of the Green's theorem, one obtains

$$\int_\Gamma \delta\sigma^*_{ji}\eta_j u_i \mathrm{d}\Gamma - \int_\Omega \delta\sigma^*_{ji,j} u_i \mathrm{d}\Omega = \int_\Gamma t_i \delta u^*_i \mathrm{d}\Gamma + \int_\Omega b_i \delta u^*_i \mathrm{d}\Omega \,. \tag{5}$$

To arrive at the above expression, one has also resorted to the identity $\sigma_{ji}\delta u^*_{i,j} \equiv u_{k,l}C_{ijkl}\,\delta u^*_{i,j} \equiv u_{k,l}\,\delta\sigma^*_{kl}$, which is not Betti's reciprocity theorem.

The conventional, collocation boundary element method may be derived from eqn (5), for fundamental solutions $\delta\sigma^*_{ij}$ and δu^*_i given as

$$\delta\sigma^*_{ij} \equiv \sigma^*_{ijm}\delta p^*_m \,, \tag{6}$$

$$\delta u^*_i = (u^*_{im} + u^r_{is}C_{sm})\delta p^*_m \,, \tag{7}$$

where u^r_{is}, for $s = 1 \ldots n^r$, are n^r rigid-body displacements that are multiplied by in principle arbitrary constants C_{sm}, and δp^*_m are arbitrary (virtual) force parameters, with m characterizing both location and direction of application of δp^*_m. Then, $\delta\sigma^*_{ijm}$ and δu^*_{im} are functions – with global support – of the coordinates and directions of δp^*_m referred to by m (the source point), as well as of the coordinates and directions referred to by i (the field point), where the effects of δp^*_m are measured.

The strength of the boundary element method stems from the fact that δp^*_m are point forces applied along Γ, just outside Ω, however infinitely close. Although $\delta\sigma^*_{ijm}$ and δu^*_{im} tend to infinity at the point of application of δp^*_m, they are analytical in Ω. For convenience, the functions $\delta\sigma^*_{ijm}$ are normalized such that, for a domain Ω_0 that contains δp^*_m, with enclosing boundary Γ_0,

$$\int_{\Omega_0} \sigma^*_{jim,j}\mathrm{d}\Omega = \int_{\Gamma_0} \sigma^*_{jim}\eta_j \mathrm{d}\Gamma \equiv -\delta_{im} \,, \tag{8}$$

where δ_{im} is a generalized Kronecker delta (equal to either 1, when i and m refer to the same degree of freedom, or 0, otherwise). According to the above definition of fundamental solution, the domain integral on the left-hand side of eqn (5) is actually evaluated as $\int_\Omega \delta\sigma^*_{ji,j} u_i \mathrm{d}\Omega = -\delta_{im} u_i \delta p^*_m \equiv -u_m \delta p^*_m$.

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Substituting for $\delta\sigma^*_{ijm}$ and δu^*_{im} in eqn (5) according to their expressions in eqns (6, 7), one obtains the modified expression of the Somigliana's identity,

$$u_{im} = \int_\Gamma t_i u^*_{im} \mathrm{d}\Gamma - \int_\Gamma \sigma^*_{jim} \eta_j u_i \mathrm{d}\Gamma + \int_\Omega b_i u^*_{im} \mathrm{d}\Omega + C_{sm}\left(\int_\Gamma t_i u^r_{is} \mathrm{d}\Gamma + \int_\Omega b_i u^r_{is} \mathrm{d}\Omega \right), \quad (9)$$

which is used to evaluate displacements u_{im} (and, subsequently, stresses) at a domain point m for prescribed forces b_i, t_i, and boundary displacements u_i. The term in brackets vanishes only if b_i and t_i are in equilibrium, which is not necessarily true when one is dealing with approximations. Then, the results are in principle influenced by some arbitrary constants C_{sm} [1].

2.3 Numerical discretization

Equation (9) is also used to evaluate the displacements u_i and the traction forces t_i as the problem's unknowns along Γ_σ and Γ_u, respectively. In fact, they may be approximated along Γ as

$$u_i = u_{in} d_n, \quad (10)$$

$$t_i = t_{i\ell} t_\ell, \quad (11)$$

where d_n, for $n = 1 \ldots n^d$, is a vector of n^d nodal displacements and u_{in} are interpolation functions with local support, usually polynomials chosen in such a way that, at the nodal points, $u_{in} \equiv \delta_{in}$. Since the traction forces t_i are surface attributes, the n^t parameters t_ℓ are also surface attributes that depend on the outward normal η_i of the boundary point at which t_ℓ is physically attached. Generally, $n^t > n^d$, as the boundary Γ may not be entirely smooth, with more than one normal at some points. The interpolation functions $t_{i\ell}$ also have local support, but are not necessarily locally expressed as u_{in} (see Section 3).

The boundary geometry is approximated from the nodal attributes using the same interpolation functions u_{in} of eqn (10), which consists in an isoparametric representation of the problem, exactly as in the finite element method.

Replacing u_i and t_i in eqn (9) with their boundary approximations given by eqns (10, 11), and applying δp^*_m at successive boundary points in such a way that $\delta p^*_m d_m$ has the meaning of virtual work, one arrives at the basic equation of the conventional, collocation boundary element method, as given in the literature,

$$\left(\int_\Gamma \sigma^*_{jim} \eta_j u_{in} \mathrm{d}\Gamma + \delta_{mn} \right) d_n = \int_\Gamma t_{i\ell} u^*_{im} \mathrm{d}\Gamma t_\ell + \int_\Omega b_i u^*_{im} \mathrm{d}\Omega + C_{sm}\left(\int_\Gamma t_{i\ell} u^r_{is} \mathrm{d}\Gamma t_\ell + \int_\Omega b_i u^r_{is} \mathrm{d}\Omega \right), \quad (12)$$

except for the term related to the constants C_{sm}, that is actually an error term. Equation (12) may be written in matrix format as

$$\mathbf{Hd} = \mathbf{Gt} + \mathbf{b} + \boldsymbol{\varepsilon}\ , \tag{13}$$

where $\mathbf{H} = [H_{mn}] \in R^{n^d \times n^d}$ is a kinematic transformation matrix, $\mathbf{G} = [G_{m\ell}] \in R^{n^d \times n^t}$ is a flexibility-type matrix and $\mathbf{b} = [b_m] \in R^{n^d}$ is a vector of nodal displacements equivalent to the applied body forces. The double-layer and single-layer potential matrices $\mathbf{H}$ and $\mathbf{G}$ comprise in their definition singular and improper integrals, respectively, when source (index *m*) and field (index either *n* or ℓ) refer to the same nodal points. Then, special care must be taken in the numerical integrations. This has been sufficiently investigated and is not the subject of the present outline. The error term $\boldsymbol{\varepsilon}$ in eqn (13), defined according to eqn (12), corresponds to residuals whose magnitude depends on the amount of rigid-body displacements that are always implicit in the fundamental solution, eqn (7), as well as on how refined is the boundary mesh, that is, how accurately the boundary traction forces, approximated according to eqn (11), are in equilibrium with the applied domain forces b_i. This vector of residuals is usually disregarded in the implementations shown in the literature [1, 2], or sometimes used as a measure of convergence of the numerical model. A consistent numerical model must take this term explicitly into account and end up with a formulation that is independent of C_{sm} instead of just disregarding it.

This specific issue has already been the subject of a theoretical investigation [3]. The main results are summarized in the following, also introducing a convenient simplification related to the particular solution term $\mathbf{b}$ of eqn (13).

The vector of residuals $\boldsymbol{\varepsilon}$ in eqn (13) may be written as

$$\boldsymbol{\varepsilon} = \mathbf{C}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}\left(\mathbf{t} - \mathbf{t}^{p}\right), \tag{14}$$

where $\mathbf{R} = [R_{\ell s}] \in R^{n^t \times n^r}$ is defined as

$$R_{\ell s} = \int_{\Gamma} t_{i\ell} u^{r}_{is} \mathrm{d}\Gamma\ , \tag{15}$$

and the product $\mathbf{R}^{\mathrm{T}}\mathbf{t}^{p}$ comes from the approximation

$$\int_{\Omega} b_i u^{r}_{is} \mathrm{d}\Omega = -\int_{\Gamma} \sigma^{p}_{ji} \eta_j u^{r}_{is} \mathrm{d}\Gamma \approx -\int_{\Gamma} t_{i\ell} u^{r}_{is} \mathrm{d}\Gamma t^{p}_{\ell}\ , \tag{16}$$

whenever a particular solution for the body force problem, as stated in eqn (1), is available. By the same token, the vector b_m of equivalent nodal displacements, introduced in eqn (13), may be approximated as developed in the following:

$$\begin{gathered} \int_{\Omega} b_i u^{*}_{im} \mathrm{d}\Omega = -\int_{\Gamma} \sigma^{p}_{ji} \eta_j u^{*}_{im} \mathrm{d}\Gamma + \int_{\Gamma} \sigma^{*}_{jim} \eta_j u^{p}_{i} \mathrm{d}\Gamma + \delta_{im} u^{p}_{i} \\ \Rightarrow \quad b_i \approx -G_{m\ell} t^{p}_{\ell} + H_{mn} d^{p}_{n}, \end{gathered} \tag{17}$$

since, for a sufficiently refined boundary mesh, the displacements u^{p}_{i} and the traction forces $t^{p}_{i} = \sigma^{p}_{ji}\eta_j$, related to an arbitrary particular solution of the nonhomogeneous governing eqn (1), whenever available [7], can be approximated accurately enough by nodal displacement and traction force parameters d_n^p and t_ℓ^p in terms of the interpolation functions of eqns (10, 11):

$$\begin{aligned} u_i^p &\approx u_{in} d_n^p, \\ \sigma_{ji}^p \eta_j &\approx t_{i\ell} t_\ell^p \quad \text{on} \quad \Gamma. \end{aligned} \tag{18}$$

Then, making use of eqns (14, 17), a convenient way of expressing eqn (13) is

$$\mathbf{H}\left(\mathbf{d}-\mathbf{d}^p\right)=\left(\mathbf{G}+\mathbf{C}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}\right)\left(\mathbf{t}-\mathbf{t}^p\right). \tag{19}$$

One identifies in eqn (14), as supported by other linear algebra manipulations [3], that the columns of the matrix $\mathbf{R}$ in eqn (19) span the space of traction forces $\left(\mathbf{t}-\mathbf{t}^p\right)$ that cannot be transformed (are not in equilibrium). Then,

$$\left(\mathbf{G}+\mathbf{C}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}\right)\mathbf{R}=\mathbf{0} \quad \Rightarrow \quad \mathbf{C}^{\mathrm{T}}=-\mathbf{G}\mathbf{R}\left(\mathbf{R}^{\mathrm{T}}\mathbf{R}\right)^{-1}, \tag{20}$$

which leads to the consistent boundary element equation

$$\mathbf{H}\left(\mathbf{d}-\mathbf{d}^p\right)=\mathbf{G}_a\left(\mathbf{t}-\mathbf{t}^p\right)\equiv\mathbf{G}\mathbf{P}_R^{\perp}\left(\mathbf{t}-\mathbf{t}^p\right), \tag{21}$$

where $\mathbf{G}_a \equiv \mathbf{G}\mathbf{P}_R^{\perp}$ is the admissible part of $\mathbf{G}$ and

$$\mathbf{P}_R^{\perp}=\mathbf{I}-\mathbf{P}_R=\mathbf{I}-\mathbf{R}\left(\mathbf{R}^{\mathrm{T}}\mathbf{R}\right)^{-1}\mathbf{R}^{\mathrm{T}} \tag{22}$$

is the orthogonal projector onto the *admissible* space of the traction forces, which comprises the subsets of traction forces that are in balance and can therefore be transformed into equivalent nodal displacements via the flexibility matrix $\mathbf{G}_a$.

2.4 Spectral assessment of the matrices involved in the BEM

Let $\mathbf{W}=\left[W_{ns}\right]\in R^{n^d\times n^r}$ be a matrix whose columns form an orthogonal basis of the nodal displacements $\mathbf{d}$ of eqn (19) related to rigid-body displacements, such that $\mathbf{W}^{\mathrm{T}}\mathbf{W}=\mathbf{I}$. Then, the rigid-body displacement functions u_{is}^r introduced in eqn (7) may be normalized in such a way that their nodal values coincide with W_{ns} at the nodal points, following that

$$u_{is}^r = u_{in} W_{ns} \quad \text{on} \quad \Gamma. \tag{23}$$

Moreover, it is sometimes advisable to think of the boundary traction forces as expressed in terms of equivalent nodal forces $\mathbf{p}=\left[p_n\right]\in R^{n^d}$ that come up from the virtual work statement

$$\delta d_m p_m = \delta d_m \int_\Gamma u_{im} t_{i\ell} \mathrm{d}\Gamma\, t_\ell$$
$$\Rightarrow \quad p_m = L_{\ell m} t_\ell \quad \text{or} \quad \mathbf{p} = \mathbf{L}^{\mathrm{T}}\mathbf{t}, \tag{24}$$

where $\mathbf{L}^{\mathrm{T}}$, as defined above, performs an equilibrium transformation.

With the definitions of $\mathbf{W}$ and $\mathbf{L}^{\mathrm{T}}$, given above, one checks the equivalence

$$\mathbf{R} \equiv \mathbf{LW}, \tag{25}$$

for $\mathbf{R}$ defined as in eqn (15), which means that, for a finite domain,

$$\mathbf{W}^{\mathrm{T}}(\mathbf{p} - \mathbf{p}^p) = \mathbf{0} \quad \Leftrightarrow \quad \mathbf{R}^{\mathrm{T}}(\mathbf{t} - \mathbf{t}^p) = \mathbf{0}. \tag{26}$$

The relations above help to shed light on the spectral properties of the matrices $\mathbf{H}$ and $\mathbf{G}_a$ of eqn (21). $\mathbf{W} = N(\mathbf{H})$ and $\mathbf{G}_a\mathbf{R} = \mathbf{0}$ is a partial consistency check. Defining $\mathbf{V}$ as the null space $\mathbf{V} = N(\mathbf{H}^{\mathrm{T}})$, one checks that $|\mathbf{V}^{\mathrm{T}}\mathbf{G}_a| \approx 0$, and not $|\mathbf{V}^{\mathrm{T}}\mathbf{G}_a| = 0$, which means that eqn (21) is not completely consistent. This is expected, as eqn (21) is obtained from a weighted-residuals statement, eqn (4), that is not variationally consistent – just compare it with eqn (3).

2.5 On the numerical solution of the problem

The inconsistent version of eqn (21),

$$\mathbf{H}(\mathbf{d} - \mathbf{d}^p) = \mathbf{G}(\mathbf{t} - \mathbf{t}^p), \tag{27}$$

obtained by neglecting the residual error $\boldsymbol{\varepsilon}$ in eqn (13), may be expressed as

$$\mathbf{Ax} = \mathbf{y}, \tag{28}$$

where the vector $\mathbf{x}$ gathers all the unknown coefficients of $\mathbf{d}$ and $\mathbf{t}$; $\mathbf{y}$ is the vector of known quantities; and the non-symmetric matrix $\mathbf{A}$ is obtained by adequately collecting the columns of $\mathbf{H}$ and $\mathbf{G}$ corresponding to the unknown coefficients [1]. Although $\mathbf{G}$ is a rectangular matrix, $\mathbf{A}$ becomes square if the boundary conditions are adequately introduced [8]. If one is lucky, $|\mathbf{V}^{\mathrm{T}}\mathbf{G}| >> 0$ and $\mathbf{A}$ is well conditioned. However, there is an in principle uncontrolled amount of rigid-body displacements implicit in the term u^*_{im} of eqn (7), so that the good conditioning of $\mathbf{A}$ cannot be assured beforehand [9].

The solution of eqn (28), if obtained from eqn (21), leads to a matrix $\mathbf{A}$ that is by construction ill conditioned. However, inclusion of the restriction $\mathbf{R}^{\mathrm{T}}(\mathbf{t} - \mathbf{t}^p) = \mathbf{0}$, eqn (26), enables the establishment of an equation system that is always well conditioned, to be solved in terms of generalized inverses. As several numerical tests have shown [3, 8], results using either $\mathbf{G}$ or $\mathbf{G}_a$ are comparable in terms of accuracy, provided that $\mathbf{G}$ does not become ill conditioned. Although the residual $\boldsymbol{\varepsilon}$ of eqn (13) tends to vanish with increasing mesh refinement, an eventual ill-conditioning of $\mathbf{G}$, once present, persists independently from mesh refinement.

2.6 Evaluation of a stiffness matrix

Instead of solving a problem in terms of the transformed system of eqn (28), one may need to obtain a formulation in terms of a stiffness-type matrix,

$$\mathbf{K}\left(\mathbf{d}-\mathbf{d}^{p}\right)=\left(\mathbf{p}-\mathbf{p}^{p}\right). \tag{29}$$

The matrix $\mathbf{K}$ is obtained by solving for $\left(\mathbf{t}-\mathbf{t}^{p}\right)$ in either eqn (21) or (27) and making use of $\mathbf{L}^{\mathrm{T}}$ defined in eqn (24). A first possibility comes from eqn (27),

$$\mathbf{K}_{inconsistent\_1}=\mathbf{L}^{\mathrm{T}}\mathbf{L}\left(\mathbf{GL}\right)^{-1}\mathbf{H}\ , \tag{30}$$

where $\mathbf{L}\left(\mathbf{GL}\right)^{-1}$ is a $\{1, 2, 3\}$-inverse of $\mathbf{G}$ ($\mathbf{GL}$ supposedly well conditioned). An alternative is to resort to the $\{1, 2, 3, 4\}$-inverse $\mathbf{G}^{\mathrm{T}}\left(\mathbf{GG}^{\mathrm{T}}\right)^{-1}$, obtaining

$$\mathbf{K}_{inconsistent\_2}=\mathbf{L}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}}\left(\mathbf{GG}^{\mathrm{T}}\right)^{-1}\mathbf{H}\ . \tag{31}$$

Either construction above leads to forces that are not in equilibrium. A conceptual improvement is

$$\mathbf{K}_{inconsistent\_3}=\mathbf{L}^{\mathrm{T}}\mathbf{P}_{R}^{\perp}\mathbf{L}\left(\mathbf{GL}\right)^{-1}\mathbf{H}\ . \tag{32}$$

However, the only fully consistent formulation stems from eqn (21) [8],

$$\mathbf{K}_{consistent}=\mathbf{L}^{\mathrm{T}}\mathbf{P}_{R}^{\perp}\mathbf{L}\left(\mathbf{G}_{a}\mathbf{L}+\mathbf{WW}^{\mathrm{T}}\right)^{-1}\mathbf{H}\ . \tag{33}$$

3 A subtle simplification and improvement

The functions u_{in} and $t_{i\ell}$ that interpolate displacements and traction forces, as introduced in eqns (10, 11) for curved, isoparametric elements, are usually polynomials of the boundary natural coordinates, a usage possibly borrowed from the finite element tradition. This is fine for the displacements (representation of rigid-body displacements is a requirement) and suited for the evaluation of the matrix $\mathbf{H}$ defined in eqns (12, 13), since the Jacobian $|J|$ of the parametric transformation cancels out in the product $\eta_j \mathrm{d}\Gamma$, whether a 2D or 3D problem is being modeled, and the integrand consists of low-order polynomials that multiply the kernel σ^{*}_{ijm}.

However, there is no justification for the traction forces to be interpolated by a polynomial along a curved boundary, since, because of the term η_j in eqn (2), the boundary forces vary according to the inverse of $|J|$ (there is not such a requirement as the representation of constant-traction forces along the boundary). Then, one suggests the replacement of the polynomials $t_{i\ell}$ in eqn (11) by (a caveat: do not use together with the quarter-point trick in fracture mechanics)

$$t_{i\ell} \leftarrow \frac{|J|_\ell}{|J|} t_{i\ell}, \tag{34}$$

where $|J|_\ell$ is the value of the Jacobian at the point characterized by the subscript ℓ. Nothing changes formally in the developments outlined for the CBEM, except that the numerical integration of the matrix $\mathbf{G}$ becomes much easier. In fact, $|J|$ cancels out in the product $t_{i\ell}\mathrm{d}\Gamma$, according to eqns (12, 13), for $t_{i\ell}$ defined as above, and the only term to be approximated by a polynomial, in the numerical integration of $\mathbf{G}$, is the kernel u^*_{im}. Moreover, $L_{\ell m} = \int_\Gamma u_{im} t_{i\ell} \mathrm{d}\Gamma$ in eqn (24) now involves only low-order polynomials in the integrand.

4 Numerical examples

Figure 1 illustrates an irregularly-shaped continuum for a set of five 2D numerical examples of potential problem, which presents the same conceptual issues as the ones of an elasticity problem. The mesh shown corresponds to examples M2_quad and M2_quadN, with a total of 62 quadratic elements and 124 nodes (46 elements along the external boundary and 16 elements modeling the hole indicated). There are also examples M1_quad and M1_quadN, with 23 quadratic elements and 46 nodes only along the external boundary (no hole is modeled), as well as example M1_lin with the same mesh as before, but comprising 46 linear elements. In the examples M2_quad and M1_quad, polynomials $t_{i\ell}$ are used, as initially proposed in eqn (11). In the examples M2_quadN and M1_quadN, one uses $t_{i\ell}$ according to eqn (34).

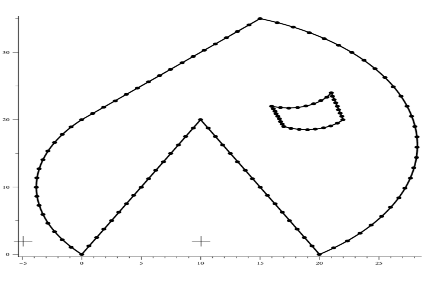

Figure 1: Discretization scheme to illustrate five numerical examples [4].

The first row of Table 1 shows the Euclidean norms of the matrix product $\left|\mathbf{K}^{\mathrm{T}}\mathbf{W}\right|$ for the examples under analysis, where $\mathbf{K}$ is the inconsistent matrix defined in eqn (30), to check that the lack of equilibrium of the ‘stiffness’ system derived from the CBEM is not so relevant for a sufficiently fine mesh, as already known in the literature. (Such an Euclidean norm for the ‘stiffness’ matrices of eqns (32, 33) is equal to zero by construction.) One also shows, in the second row, the Euclidean norms of the matrix product $\left|\mathbf{G}_a^{\mathrm{T}}\mathbf{V}\right|$, for $\mathbf{G}_a$ as in eqn (21), to attest the comment made in the paragraph after eqn (26) as well as to corroborate the outlines of Sections 2.5 and 2.6.

Table 1: Euclidean norms of two matrix products.

	M1_lin	M1_quad	M1_quadN	M2_quad	M2_quadN
$\left\|\mathbf{K}^{\mathrm{T}}\mathbf{W}\right\|$	.688e-3	.167e-2	.168e-2	.798e-3	.799e-3
$\left\|\mathbf{G}_a^{\mathrm{T}}\mathbf{V}\right\|$	.545e-2	.442e-2	.448e-2	.232e-2	.233e-2

One also runs a series of patch tests for a total of ten potential fields applied to the models, eight of them, numbered 1 to 8, are listed in the vector $\left\langle x \quad y \quad xy \quad x^2-y^2 \quad x^3-3xy^2 \quad -3x^2y+y^3 \quad x^4+y^4-6x^2y^2 \quad x^3y-xy^3 \right\rangle$, besides potential fields number 9 and 10 that correspond to $\ln(r)/2\pi$, where r is the distance to the source points (-5, 2) and (10, 2) marked as two crosses in Figure 1. The accuracy of the solution with the inconsistent matrix $\mathbf{G}$, as given in eqn (27), is assessed in the first graphic of Figure 2, for all ten potential fields, with corresponding values of $\mathbf{d}$ and $\mathbf{t}$ evaluated analytically. The results using $\mathbf{G}_a$, as in eqn (21), are almost indistinguishable from the ones with $\mathbf{G}$.

The second graphic in Figure 2 assesses the accuracy of using $\mathbf{L}^{\mathrm{T}}$, as given in eqn (24), to evaluate equivalent nodal gradients, for $\mathbf{p}$ directly integrated as $p_m = \int_\Gamma u_{im} t_i \mathrm{d}\Gamma$. The last graphic assesses the accuracy of the ‘stiffness’ system of eqn (29), where $\mathbf{K}$ is the inconsistent matrix defined in eqn (30). Results using the alternative definitions of $\mathbf{K}$ in eqns (31-33) are almost indistinguishable from these ones. The generalized inverses used to arrive at $\mathbf{K}$ in eqns (30-33) deserve further numerical investigation. In fact, space restrictions of the manuscript prevent the elaboration of many interesting conclusions from these simple examples and graphics.

The curved parts of the boundaries are very deleterious to the numerical accuracy, particularly affecting the error norm $\left|\mathbf{L}^{\mathrm{T}}\mathbf{t}-\mathbf{p}\right|$. One checks that the suggestion expressed in eqn (34) significantly improves the problems’ response to constant gradients, in terms of both $\left|\mathbf{L}^{\mathrm{T}}\mathbf{t}-\mathbf{p}\right|$ and $\left|\mathbf{Hd}-\mathbf{Gt}\right|$, but does not lead to perceptible improvements when testing for higher order gradients. However, the simplification achieved with eqn (34) regarding numerical implementation is per se an improvement. Owing to the errors introduced by the approximation $\mathbf{p} \approx \mathbf{L}^{\mathrm{T}}\mathbf{t}$ for curved boundaries, results in terms of $|\mathbf{Hd} - \mathbf{Gt}|$ are in general significantly better than in terms of $|\mathbf{Kd} - \mathbf{p}|$ – observe that this is not true for linear elements.

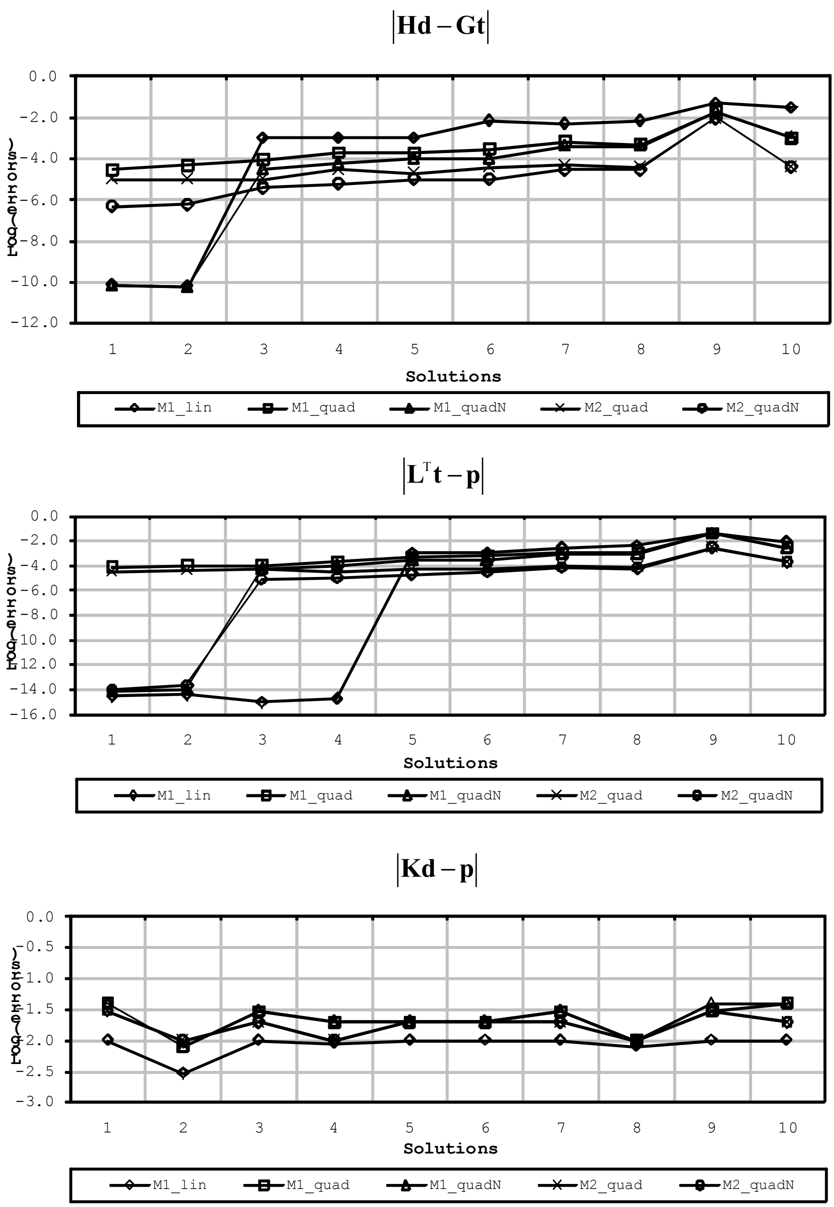

Figure 2: Error norms for three matrix equations of the potential problem.

5 Conclusions

As outlined, the matrix **G** is actually rectangular. One suggests a subtle improvement and simplification of the boundary representation of traction forces. The issue of continuity or discontinuity of such forces at corner points is simply non-existent in the frame of a consistent formulation. The system matrix is always well conditioned for an adequately formulated problem. An extended version of this manuscript is being prepared for publication, in which spectral properties and generalized-inverse concepts of the matrices are further explored.

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