Three-dimensional steady thermal stress analysis by triple-reciprocity BEM

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Abstract

Steady thermal stress problems without heat generation can be solved easily by the boundary element method. However, for the case with arbitrary heat generation, the domain integral is necessary. In this paper, it is shown that the problems of three-dimensional steady thermal stress with heat generation can be solved approximately without the domain integral by the triple-reciprocity boundary element method. In this method, an arbitrary distribution of heat generation is interpolated by boundary integral equations. In order to solve the problem, the values of heat generation at internal points and on the boundary are used.
Keywords: Thermal stress, triple-reciprocity method, harmonic function, boundary element method.

1 Introduction

The steady thermal stress problem without heat generation can be solved easily by the boundary-element method (BEM). When analysis of thermal stress under arbitrary heat generation within the domain is carried out by the BEM, generally the domain integral is necessary. By this method, however, the merit of BEM for the simple preparation of data is lost. Several other methods have been considered. Nowak and coworkers have proposed the multiple-reciprocity method [1]. Ochiai et al. have proposed an approximate method using the cells of boundary type [2].

In the conventional multiple-reciprocity method, heat generation must be given analytically and the analytical derivation of heat generation on the boundary is necessary. Fundamental solutions of higher order are used to make the solution converge for some problems. Accordingly, the conventional

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multiple-reciprocity method is not suitable for practical steady thermal stress analyses with arbitrary internal heat generation in the domain. On the other hand, Ochiai and coworkers have proposed the improved multiple-reciprocity BEM for the steady heat conduction problem, steady thermal stress problem and elastoplasticity problem [3-6].

It is difficult to understand the improved multiple-reciprocity BEM. In this work, the thermal stress problem is solved by the triple-reciprocity BEM, which is derived from the improved multiple-reciprocity BEM [4]. Using this method, a highly accurate solution may be obtained solely by using the fundamental solutions of lower order without the need for data preparing. In this method, heat generation distribution is interpolated using the boundary integral equations [3-6]. Point and line heat sources are easily treated by the conventional BEM. Therefore, an arbitrary distributed heat source is examined in this paper.

In this paper, the three-dimensional thermal stress problem is solved by the triple-reciprocity boundary elements method. In this method, a polyharmonic function with volume distribution is used in order to obtain an exact solution. Even if the distribution of heat generation is complicated, the exact solution can be obtained. The use of a polyharmonic function with volume distribution reduces the CPU time.

2 Basic equations

2.1 Heat conduction

Point and line heat sources can easily be treated by the conventional BEM. In this study an arbitrarily distributed heat source W_1^S is treated. In steady heat conduction problems, the temperature T under an arbitrarily distributed heat source W_1^S is obtained by solving the following equation:

$$\nabla^2 T = \frac{-W_1^S}{\lambda} \tag{1}$$

where λ is thermal conductivity. Denoting heat generation by $W_1^S(q)$, the boundary integral equation for the temperature in the case of steady heat conduction problems is given by [7]

$$CT(P) = \int_\Gamma \{T_1(P,Q)\frac{\partial T(Q)}{\partial n} - \frac{\partial T_1(P,Q)}{\partial n}T(Q)\}d\Gamma(Q) + \lambda^{-1}\int_\Omega T_1(p,q)W_1^S(q)d\Omega \tag{2}$$

where C=0.5 on the smooth boundary and C=1 in the domain. The notations Γ and Ω represent the boundary and domain, respectively. The notations p and q become P and Q on the boundary. In the case of three-dimensional problems, the fundamental solution $T_1(p, q)$ in Eq.(2) for the steady temperature analyses and its normal derivative are given by

$$T_1(p,q) = \frac{1}{4\pi r} \tag{3}$$

$$\frac{\partial T_1(p,Q)}{\partial n} = \frac{-1}{4\pi r^2}\frac{\partial r}{\partial n} \tag{4}$$

where r is the distance between the observation point p and the loading point q. As shown in Eq.(2), when there exists arbitrary heat generation $W_1^S(q)$ in the domain, the domain integral is necessary. Therefore, the triple-reciprocity BEM [5-8] is used.

2.2 Interpolation

An interpolation method for heat generation is shown using boundary integral equations in order to avoid internal cells. The distribution of heat generation W_1^S is given by

$$\nabla^2 W_1^S = -W_2^S \tag{5}$$

$$\nabla^2 W_2^S = -\sum_{m=1}^{M} W_{3A(m)}^P \tag{6}$$

From Eqs.(5) and (6), the following equation can be obtained.

$$\nabla^4 W_1^S = \sum_{m=1}^{M} W_{3A(m)}^P \tag{7}$$

where the function W_{3A}^P expresses a state of a uniformly distributed polyharmonic function in a spherical region with radius A. In order to solve Eqs. (5) and (6), a polyharmonic function is introduced. Polyharmonic function T_f is defined as

$$\nabla^2 T_{f+1} = T_f \tag{8}$$

Therefore, polyharmonic function T_f is determined as

$$T_f = \int \frac{1}{r^2}[\int r^2 T_{f-1} dr]dr \tag{9}$$

Three-dimensional polyharmonic function T_f and its normal derivatives are given by

$$T_f(P,Q) = \frac{r^{2f-3}}{4\pi(2f-2)!} \tag{10}$$

$$\frac{\partial T_f}{\partial n} = \frac{(2f-3)r^{2f-4}}{4\pi(2f-2)!}\frac{\partial r}{\partial n} \tag{11}$$

Figure 1 shows the shape of polyharmonic functions, and bi-harmonic function T_2 is not smooth at r=0. In the three-dimensional case, a smooth interpolation cannot be obtained solely by bi-harmonic function T_2. In order to obtain a smooth interpolation, the polyharmonic function with volume distribution T_{2A} is introduced. A polyharmonic function with volume distribution T_{fA} as shown in Fig.2 is defined as [6]

$$T_{fA} = \int_{\Omega_A} T_f(P,q)d\Omega_A = \int_0^A \int_S T_f(P,q)dSda = \int_0^A \int_0^{2\pi} \int_0^{\pi} T_f(P,q)a^2 \sin\theta d\theta d\phi da \tag{12}$$

where Ω_A is a spherical region with radius A, and S is the surface of spherical shell with radius a. The function T_{fA} can be easily obtained using the relationships $r^2 = R^2 + a^2 - 2aR\cos\theta$ and $dr = aR\sin\theta d\theta$ as shown in Fig.2. This function is written by using r instead of R similarly to that in Eqs. (10) and (11), though the function from Eq. (12) is the function of R. The newly defined function T_{fA} can be explicitly shown as

$$T_{1A} = \frac{A^3}{3r}, \qquad r>A \tag{13}$$

$$T_{1A} = \frac{3A^2 - r^2}{6}, \qquad r \leqq A \tag{14}$$

$$T_{2A} = \frac{A^3}{6r}\left(r^2 + \frac{A^2}{5}\right), \qquad r>A \tag{15}$$

$$T_{2A} = -\frac{r^4 - 10r^2A^2 - 15A^4}{120}, \qquad r \leqq A \tag{16}$$

$$T_{3A} = \frac{A^3(35r^4 + 42r^2A^2 + 3A^4)}{2520r}, \qquad r>A \tag{17}$$

$$T_{3A} = \frac{-r^6 + 21r^4A^2 + 105r^2A^4 + 35A^6}{5040}, \qquad r \leqq A \tag{18}$$

$$T_{4A} = \frac{A^3(21r^6 + 63r^4A^2 + 27r^2A^4 + A^6)}{45360r}, \qquad r>A \tag{19}$$

$$T_{4A} = \frac{-r^8 + 36r^6A^2 + 378r^4A^4 + 420r^2A^6 + 63A^8}{362880}. \quad r \leqq A \tag{20}$$

Let the number of W_{3A}^{P} be M. Using Green's theorem two times and Eq.(8), Eqs.(5) and (6) become

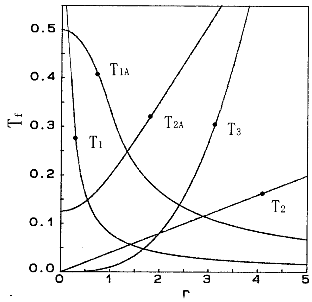


Figure 1: Functions (T_f, T_{fA}).

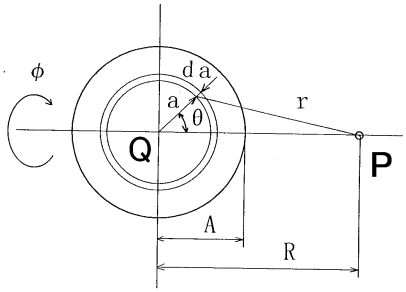


Figure 2: Notations for polyharmonic function with volume distribution.

$$CW^S{}_1(P) = \sum_{f=1}^{2}(-1)^f \int_\Gamma \{T_f(P,Q)\frac{\partial W_f^S(Q)}{\partial n} - \frac{\partial T_f(P,Q)}{\partial n}W_f^S(Q)\}d\Gamma(Q)$$

$$-\sum_{m=1}^{M} T_{2A}(P,q)W_{3A(m)}^{P}(q) \tag{21}$$

Moreover, W_2^S in Eq. (6) is similarly given by

$$CW^S{}_2(P) = \int_\Gamma \{T_1(P,Q)\frac{\partial W_2^S(Q)}{\partial n} - \frac{\partial T_1(P,Q)}{\partial n}W_2^S(Q)\}d\Gamma(Q)$$

$$+\sum_{m=1}^{M} T_{1A}(P,q)W_{3A(m)}^{P}(q) \tag{22}$$

From Eqs. (21) and (22), the unknown point load $W_{3A}^{P}(q)$ and ∂W_3^S (Q)/∂n can be obtained [5,6]. Using Green's theorem three times and Eqs.(5), (6) and (8), Eq.(2) becomes

$$CT(P) = \int_{\Gamma} \{T_1(P,Q)\frac{\partial T(Q)}{\partial n} - \frac{\partial T_1(P,Q)}{\partial n}T(Q)\}d\Gamma(Q)$$
$$-\lambda^{-1}\sum_{f=1}^{2} \int_{\Gamma} \{T_{f+1}(P,Q)\frac{\partial W_f^S(Q)}{\partial n} - \frac{\partial T_{f+1}(P,Q)}{\partial n}W_f^S(Q)\}d\Gamma(Q)$$
$$+\lambda^{-1}\sum_{m=1}^{M} T_{3A}(P,q)W_{3A(m)}^{P}(q) \tag{23}$$

2.3 Thermal stress

Next, in order to obtain the thermal stresses, let us consider the thermoelastic displacement potential Φ given by [2]

$$C\Phi(P) = \int_{\Gamma} \{\frac{\partial T\ (Q)}{\partial n}\phi_1(P,Q) - T\ (Q)\frac{\partial \phi_1(Q,P)}{\partial n}\}d\Gamma(Q) + \lambda^{-1}\int_{\Omega} \phi_1(P,q)W_1^S(q)d\Omega \tag{24}$$

where

$$\phi_1(P,Q) = m_0 T_2(P,Q) = \frac{m_0 r}{8\pi} \tag{25}$$

Denoting Poisson's ratio by ν and the coefficient of linear thermal expansion by α, m_0 is given by $m_0=(1+\nu)\alpha/(1-\nu)$. Now, let us introduce the high-order function ϕ_f defined by

$$\nabla^2 \phi_{f+1} = \phi_f \tag{26}$$

Then, using Eqs.(5), (6) and (8) and Green's theorem two times, Eq.(24) can be written as

$$C\Phi(P) = \int_{\Gamma} \{\frac{\partial T\ (Q)}{\partial n}\phi_1(P,Q) - T\ (Q)\frac{\partial \phi_1(P,Q)}{\partial n}\}d\Gamma(Q)$$
$$-\lambda^{-1}\sum_{f=1}^{2} \int_{\Gamma} \{\phi_{f+1}(P,Q)\frac{\partial W_f^S(Q)}{\partial n} - \frac{\partial \phi_{f+1}(P,Q)}{\partial n}W_f^S(Q)\}d\Gamma(Q)$$
$$+\lambda^{-1}\sum_{m=1}^{M} \phi_{3A}(P,q)W_{3A(m)}^{P}(q) \tag{27}$$

where

$$\phi_f(P,q) = m_0 T_{f+1}(P,q) \tag{28}$$

$$\phi_{fA}(P,q) = m_0 T_{f+1A}(P,q) \tag{29}$$

In the infinite space, the displacement $\bar{u}_i$ and stress $\bar{\sigma}_{ij}$ are obtained by [8]

$$\bar{u}_i = \frac{\partial \Phi}{\partial x_i} \tag{30}$$

$$\bar{\sigma}_{ij} = 2G\left(\frac{\partial^2 \Phi}{\partial x_i \partial x_j} - \delta_{ij} m_0 T\right) \tag{31}$$

where G is the shearing modulus.

Using the relationship between the thermoelastic displacement potential and the displacement, the boundary integral representation for the displacement is obtained as [2, 8]

$$\begin{aligned} C_{ij} u_j(P) = &\int_\Gamma \{u_{ij}(P,Q) p_j(Q) - p_{ij}(P,Q) u_j(Q)\} d\Gamma(Q) \\ &+ \int_\Gamma \{T(Q) \frac{\partial u_i^{(1)}(P,Q)}{\partial n} - \frac{\partial T(Q)}{\partial n} u_i^{(1)}(P,Q)\} d\Gamma(Q) \\ &+ \lambda^{-1} \sum_{f=1}^{2} \int_\Gamma \{W_{f+1}^S(Q) \frac{\partial u_i^{(f+1)}(P,Q)}{\partial n} - \frac{\partial W_{f+1}^S(Q)}{\partial n} u_i^{(f+1)}(P,Q)\} d\Gamma(Q) \\ &+ \lambda^{-1} \sum_{m=1}^{M} u_i^{(3)A}(P,q) W_{3A(m)}^P(q) \end{aligned} \tag{32}$$

where $u_{ij}(P,Q)$ and $p_{ij}(P,Q)$ are

$$u_{ij}(P,Q) = \frac{1}{16\pi G(1-\nu) r}[(3-4\nu)\delta_{ij} + r_{,i} r_{,j}] \tag{33}$$

$$p_{ij}(P,Q) = \frac{-1}{8\pi G(1-\nu) r^2}\left[\frac{\partial r}{\partial n}\{(1-2\nu)\delta_{ij} + 3 r_{,i} r_{,j}\} - (1-2\nu)(r_{,i} n_j - r_{,j} n_i)\right] \tag{34}$$

In Eqs.(33) and (34), $r_{,i} = \partial r / \partial x_i$ and n_i is the unit normal component. Using Eq.(27) and (30), $u_i^{[f]}$ and $u_i^{[3]A}$ are given by

$$u_i^{(f)} = \frac{m_0 (2f-1) r_{,i} r^{2f-2}}{4\pi (2f)!} \tag{35}$$

$$\frac{\partial u_i^{(f)}}{\partial n} = \frac{m_0(2f-1)r^{2f-3}}{4\pi(2f)!}\left[n_i + (2f-3)r,_i \frac{\partial r}{\partial n}\right] \tag{36}$$

$$u_i^{(3)A} = \frac{m_0 A^3 r,_i (105r^6 + 189r^4 A^2 + 27r^2 A^4 - A^6)}{45360r^2} \quad , \quad r>A \tag{37}$$

$$u_i^{(3)A} = \frac{m_0 r r,_i (-r^6 + 27r^4 A^2 + 189r^2 A^4 + 105A^6)}{45360} \quad , \quad r<A \tag{38}$$

The boundary integral representation of stresses in the domain is given by

$$\begin{aligned}\sigma_{ij}(p) = &\int_\Gamma \{D_{ijk}(p,Q)p_k(Q) - S_{ijk}(p,Q)u_k(Q)\}d\Gamma(Q) \\ &+ \int_\Gamma \{T(Q)\frac{\partial \sigma_{ij}^{(1)}(p,Q)}{\partial n} - \frac{\partial T(Q)}{\partial n}\sigma_{ij}^{(1)}(p,Q)\}d\Gamma(Q) \\ &+ \lambda^{-1}\sum_{f=1}^{2}\int_\Gamma \{W_{f+1}^S(Q)\frac{\partial \sigma_{ij}^{(f+1)}(p,Q)}{\partial n} - \frac{\partial W_{f+1}^S(Q)}{\partial n}\sigma_{ij}^{(f+1)}(p,Q)\}d\Gamma(Q) \\ &+ \lambda^{-1}\sum_{m=1}^{M}\sigma_{ij}^{(3)A}(p,q)W_{3A(m)}^P(q)\end{aligned} \tag{39}$$

where

$$D_{ijk} = \left[(1-2\nu)\{\delta_{ki}r,_j + \delta_{kj}r,_i - \delta_{ij}r,_k\} + 3r,_i r,_j r,_k\right]\frac{1}{8\pi(1-\nu)r^2} \tag{40}$$

$$\begin{aligned}S_{ijk} = &\frac{G}{r^3}\{3\frac{\partial r}{\partial n}[(1-2\nu)\delta_{ij}r,_k + \nu(\delta_{kj}r,_i + \delta_{ik}r,_j) - 5r,_i r,_j r,_k] + 3\nu(n_i r,_j r,_k + n_j r,_i r,_k) \\ &+ (1-2\nu)(3n_k r,_i r,_j + n_j\delta_{ik} + n_i\delta_{jk}) - (1-4\nu)n_k\delta_{ij}\}\frac{1}{4\pi(1-\nu)}\end{aligned} \tag{41}$$

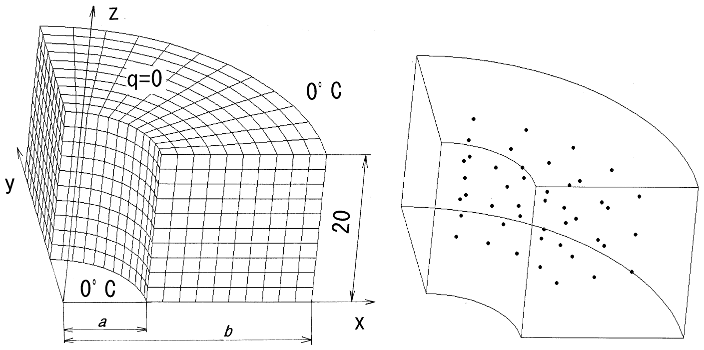
Using Eq.(27) and (31), functions $\sigma_{ij}^{(f)}$, $\partial\sigma_{ij}^{(f)}/\partial n$ and $\sigma_{ij}^{(f)A}$ in Eq.(39) are given by

$$\sigma_{ij}^{(f)} = \frac{Gm_0(2f-1)r^{2f-3}}{2\pi(2f)!}\left[-(2f-1)\delta_{ij} + (2f-3)r,_i r,_j\right] \tag{42}$$

$$\frac{\partial \sigma_{ij}^{(f)}}{\partial n} = \frac{Gm_0(2f-1)r^{2f-4}}{2\pi(2f)!}\left[r,_j n_i + r,_i n_j - (2f-1)\frac{\partial r}{\partial n}\delta_{ij} + (2f-5)r,_i r,_j \frac{\partial r}{\partial n}\right] \tag{43}$$

$$\sigma_{ij}^{(3)A} = \frac{Gm_0 A^3}{22680 r^3}[-\delta_{ij}(525r^6 + 567r^4A^2 + 27r^2A^4 + A^6)$$
$$+3(105r^6 + 63r^4A^2 - 9r^2A^4 + A^6)r,_i r,_j\,] \quad r>A \qquad (44)$$

$$\sigma_{ij}^{(3)A} = \frac{Gm_0}{11340}\left[\delta_{ij}(4r^6 - 81r^4A^2 - 378r^2A^4 - 105A^6) + 3r^2(-r^4 + 18r^2A^2 + 63A^4)r,_i r,_j\right]$$
$$r \leqq A \qquad (45)$$



(a) Boundary elements (b) Internal points

Figure 3: Cylinder with heat generation (quarter-region).

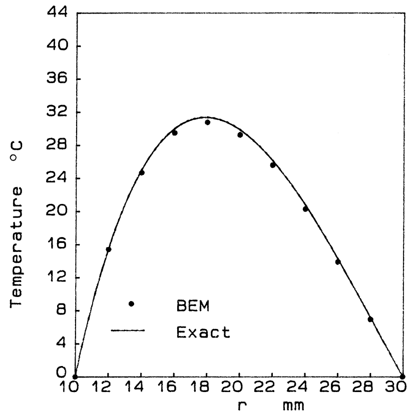


Figure 4: Temperature distribution on cylinder.

3 Numerical examples

In order to ensure the accuracy of the present method, the thermal stresses in a circular cylinder, the inner and the outer radius of which are *a* and *b* as shown in Fig.3, are obtained under heat generation:

$$W = \frac{W_0(b^2 - r^2)}{b^2 - a^2} \quad (46)$$

where r is the radial coordinate and a=10 mm and b=30 mm. The internal and outer pressures are zero. Moreover, W_0/λ=1 °Cmm^{-2} is assumed. The two-dimensional state, in which there is no heat flow in the z direction, is assumed. The upper and lower surfaces of the cylinder are restricted in the z-direction. The temperature at r=a and b is 0 ℃. Young's modulus E, Poisson's ratio ν and the coefficient of linear thermal expansion α are assumed to be 210 GPa, 0.3 and 11×10^{-6} K^{-1}, respectively. The number of discretized constant boundary element is 680, and the internal points is 45, as shown in Fig.3(a) and (b). The temperature distribution is given in Fig.4, where the solid lines show the exact solutions. Figure 5 shows the radial and circumferential thermal stress distributions with an exact solution. Radius A_i of spherical region in Eq. (12) at internal points p_i is obtained by

$$A_i = \frac{\min_{j=1,2,..,M}[r(p_i, q_j)]}{2} \quad (47)$$

where $r(p_i,q_j)$ is the distance between the internal points p_i and q_i for interpolation.

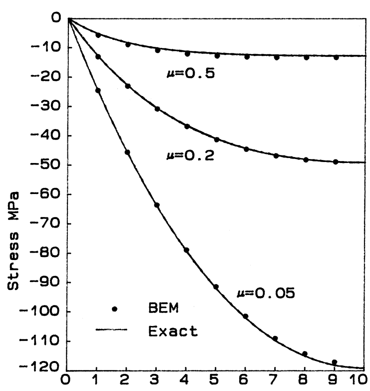
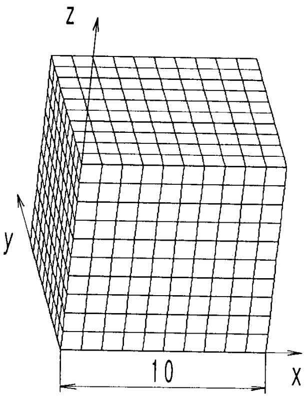


Figure 5: Stress distribution on cylinder.

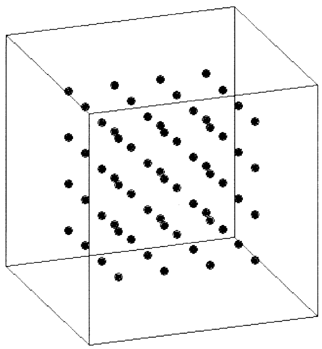
Next, the thermal stresses in a cube, of length L=10 mm, as shown in Fig.6, are obtained with the heat generation.

$$W = W_0 \exp(-\mu x) \quad (48)$$

The temperature at x=0 is 0°C, and is adiabatic at y=0, z=0, y=10, z=10 and x=10.



(a) Boundary elements



(b) Internal points

Figure 6: Cube with heat generation.

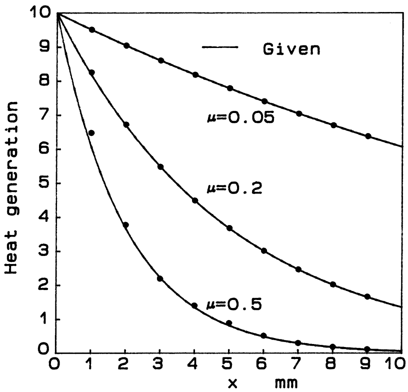


Figure 7: Interpolation of heat generation.

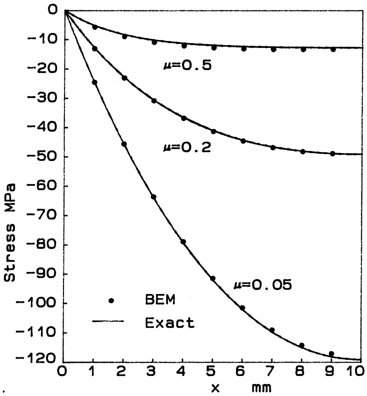


Figure 8: Stress distribution on in cube cube (σ_{yy}).

The surfaces at x=0mm and x=10mm are free, and the displacements of surfaces at y=z=0mm and y=z=10 mm are 0. Heat generation W_0/λ=1 ℃mm^{-2} is assumed. The number of discretized constant boundary element is 600, and the internal points for interpolation is 64, as shown in Fig.6(a) and (b). The other conditions are the same as in Fig.3. Figure 7 shows comparison with the interpolated value and given distributions of heat generations at μ=0.05, 0.2 and 0.5 mm^{-1}. Figure 8 shows thermal stress distribution (σ_{yy}) at y=z=5 mm.

4 Conclusion

It has been shown that it is possible to express the distributions of heat generation using integral equations. It has also been shown that steady thermal stress analysis of higher accuracy using the boundary integral is possible by the triple-reciprocity BEM even in the case of arbitrary distributions of heat generation. Accordingly, solely by adding the data of the values at internal points and on the boundary for the distributions of heat generation, the analysis of steady thermal stress with heat generation has become possible using only few data values. In the presented formulation, the thermoelastic displace potential is very effective method.

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