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A study on the source points locations in the method of fundamental solution

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Abstract

This paper demonstrates that the placement of sources in the MFS can be chosen arbitrary. Unlike the popular belief, which recommends placing them in a zone within a certain limit from the boundary, it was found that placing them at the infinity (numerically: very large distances) might provide the most stable results. It has been demonstrated also that the number of sources is totally independent of the number of boundary points. It is recommended that the number of sources should be less than that of the boundary points.

1 Introduction

The Method of Fundamental Solution (MFS), a discrete form of the indirect BEM, has proved to be a fast and easy way to implement for solving a wide variety of problems, such as potential [1], biharmonic [2], acoustics [3] and elastosatics problems [4-5].
The main advantage of this method is that it doesn't need meshing of both the problem domain and boundary. It can be also used to compute values very near to, or even on the boundaries, and there is no singularity in the solution, as the sources are always placed outside the domain. On the other hand, the solution of the system of equation is always ill-conditioned. The best approximation for the boundary conditions and the geometry can be represented only by stepwise constant distribution.

The MFS is based on getting the solution of the considered problem by distributing the fundamental solution, or its derivatives via a number of sources located outside the problem domain. The location of these sources was a subject of research to most of researchers in the last 30 years. Some researchers suggested that the sources should be placed at an offset distance from the boundary [2], while others found they are better to be distributed around the domain on a circle of diameter 2 to 4 times of the largest dimension of the domain [1]. Refinement procedures to find the optimum placement of the

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sources had been also considered via least squares [6], or via random search as in the simulated annealing algorithm [7].

In this paper, it will be shown that the accuracy of the results is slightly affected by the location of sources. It is important to note that accurate manipulation of the solution procedure is extremely important to deal with the ill-conditioning nature of the solution procedures.

2 Method of Fundamental Solution and Influence Coefficients

For a potential u in a homogeneous isotropic medium satisfies the Laplace equation:

$$\nabla^2 u = 0 \tag{1}$$

Using the MFS, the potential at any point X can be represented by the following discrete summation:

$$u(\mathbf{X}) = \sum_{i=1}^{N} \left(u^*(\mathbf{X}, \xi_i) \cdot \varphi(\xi_i) \right) \tag{2}$$

where,

$u(\mathbf{X})$ is the potential at point **X**,

$u^*(\mathbf{X}, \xi_i)$ is the potential at point **X** due to the influence of a source at ξ_i (the fundamental solution kernel),

$\varphi(\xi_i)$ is an unknown fictitious flux, and

N is the number of sources.

Similar equation can be obtained for the flux (potential gradient) distribution by differentiating equation (2) with respect to the outward normal at the point X (when X is on the boundary)

$$q(\mathbf{X}) = \sum_{i=1}^{N} \left(q^*(\mathbf{X}, \xi_i) \cdot \varphi(\xi_i) \right) \tag{3}$$

where,

$q(\mathbf{X})$ is the flux at point **X**

$q^*(\mathbf{X}, \xi_i)$ is the potential at point **X** due to the effect of source ξ_i

In order to solve the problem, the values of $\varphi(\xi_i)$ - where i=1 to N – should be determined first. If equation (2) and (3) is applied to all boundary points where the boundary conditions are known, the rectangular system of equations shown in equation (4) is obtained. It should be noted that equation (2) is used when the prescribed boundary condition at the considered point **X** is the potential u , and (3) when the prescribed one is the flux q .

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$$\begin{pmatrix} uq^*(X_1,\xi_1) & uq^*(X_1,\xi_2) & \cdots & uq^*(X_1,\xi_N) \\ uq^*(X_2,\xi_1) & & & \\ \vdots & & & \vdots \\ uq^*(X_M,\xi_1) & & \cdots & uq^*(X_M,\xi_N) \end{pmatrix} \begin{pmatrix} \varphi(\xi_1) \\ \varphi(\xi_2) \\ \vdots \\ \varphi(\xi_N) \end{pmatrix} = \begin{pmatrix} uq(X_1) \\ uq(X_2) \\ \vdots \\ uq(X_M) \end{pmatrix} \quad (4)$$

$$A\,x = B$$

where,

$uq^*(\mathbf{X}_j,\xi_i)$ is the potential u^*, or the flux q^*, at point $\mathbf{X}_j$ due to the effect of source ξ_i

$uq(\mathbf{X}_j)$ is the prescribed potential u, or the flux q, at point $\mathbf{X}_j$ (obtained from the boundary conditions)

After solving the former system of equations, the values of $\varphi(\xi_i)$ can be determined. Then, using equations (2) and (3) another time to compute the values of u or q at any point $\mathbf{X}$.
The system of equations (4) is usually ill-conditioned, as mentioned by Chen [8], however, this ill-conditioning can be cancelled out during the computation of the values of u and q, provided that the used machine precision is enough to preserve all significant digits. Most of researchers use the FORTRAN 'double precision' (or similar in C) in programming such problems. As this level of precision was not enough to track such ill-conditioning, the main motive of research in the literature (as mentioned in the introduction section) was to adjust the location of sources to decrease ill-conditioning as possible, in order to make it within the capturing level of the FORTRAN double precision. In this paper, advanced numerical software, such as the Matlab package [9], is used to improve the accuracy of arithmetic operations and to preserve all significant digits in order to have more accurate modeling.

3 Numerical examples

In this section, two examples are presented for potential problems. They are solved using the MFS. The results are compared to their analytical values. The sources location, the number of sources and the number of boundary points are taken as the main variables, and their effect on the resulting error is demonstrated and discussed. The studied ranges for these variables are:

- sources location radius: vary from 2 to 1005,
- number of boundary points: vary from 12 to 500, and
- number of sources: vary from 2, 4 or 5 to the same number as that of the boundary points.

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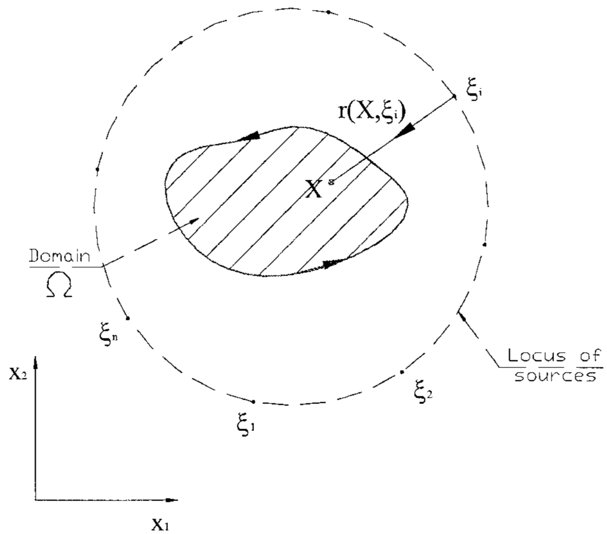

Fig.(1) – The location of sources ξ_i relative to the domain Ω

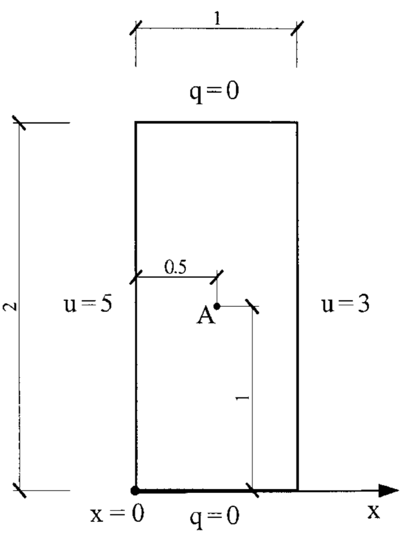

Fig.(2) – Geometry and boundary conditions for example 1.

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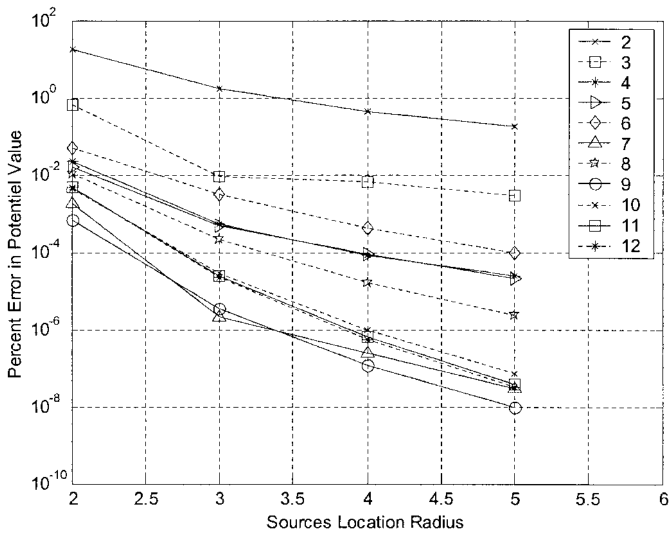

Fig.(3) – Example 1: Percent error in potential for 12 boundary points, and 2 to 12 sources.
The studied sources location radius ranging from 2 to 5

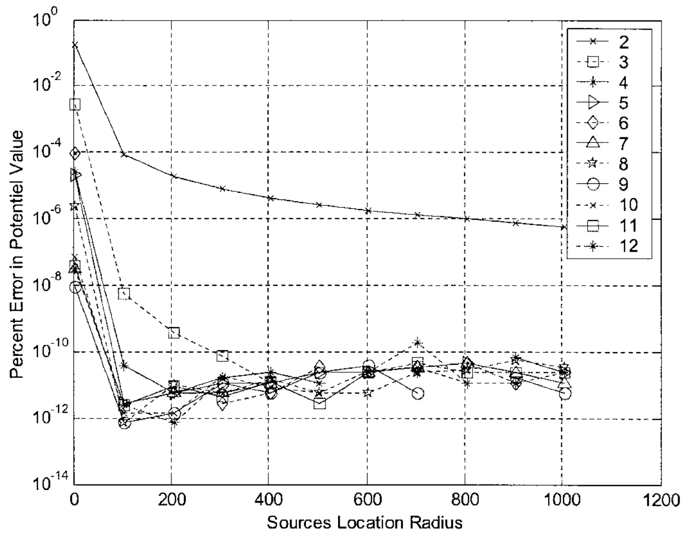

Fig.(4) – Example 1: Percent error in potential for 12 boundary points, and 2 to 12 sources.
The studied sources location radius ranging from 5 to 1005

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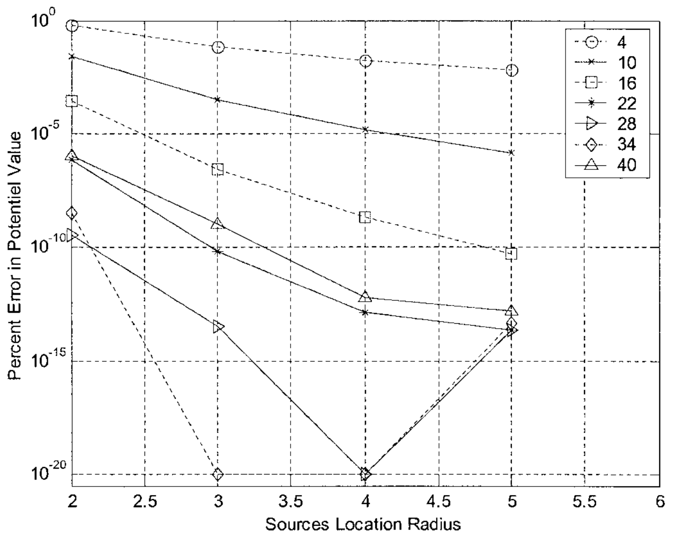

Fig.(5) – Example 1: Percent error in potential for 40 boundary points, and 4 to 40 sources.
The studied sources location radius ranging from 2 to 5

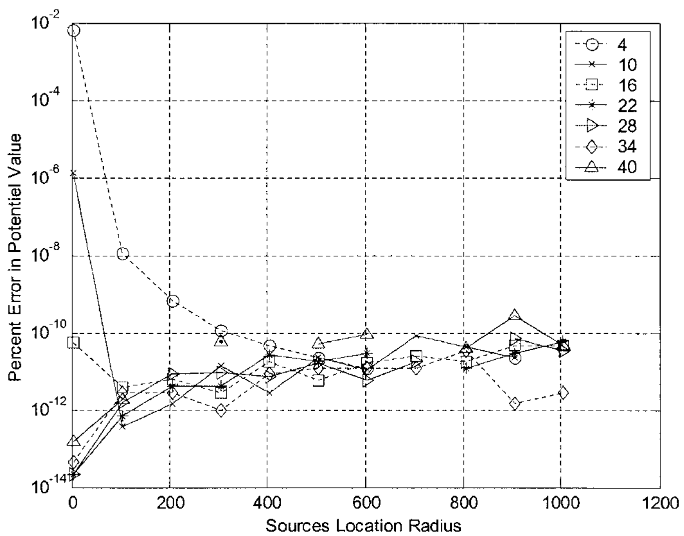

Fig.(6) – Example 1: Percent error in potential for 40 boundary points, and 4 to 40 sources.
The studied sources location radius ranging from 5 to 1005

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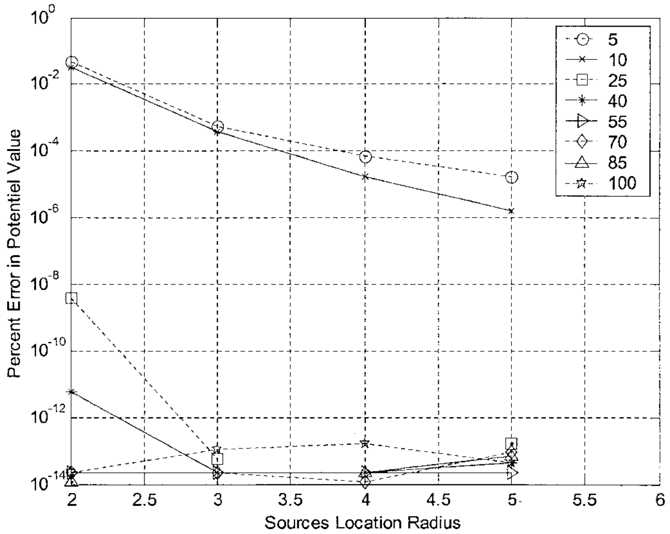

Fig.(7) – Example 1: Percent error in potential for 100 boundary points, and 5 to 100 sources.
The studied sources location radius ranging from 2 to 5

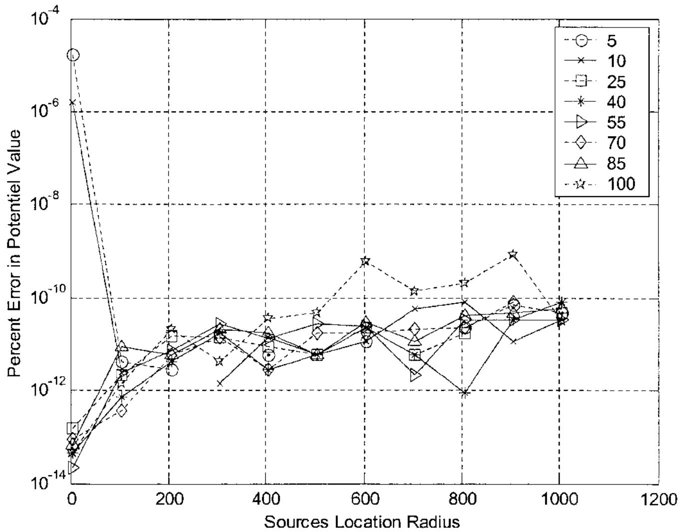

Fig.(8) – Example 1: Percent error in potential for 100 boundary points, and 5 to 100 sources.
The studied sources location radius ranging from 5 to 1005

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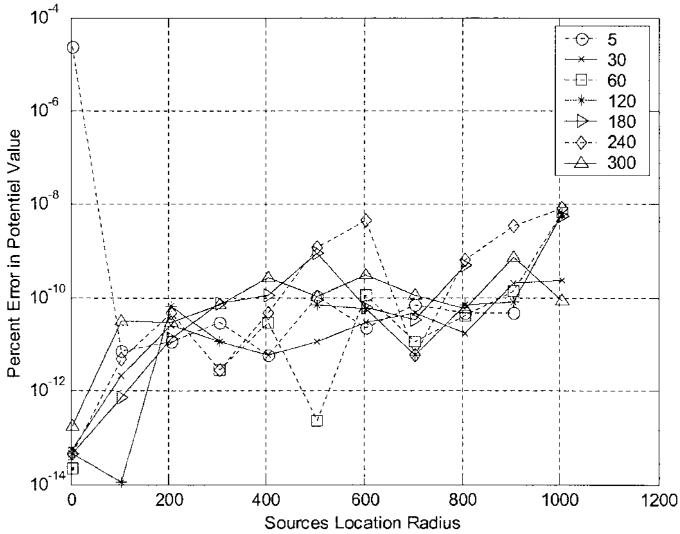

Fig.(9) – Example 1: Percent error in potential for 300 boundary points, and 5 to 300 sources.
The studied sources location radius ranging from 5 to 1005

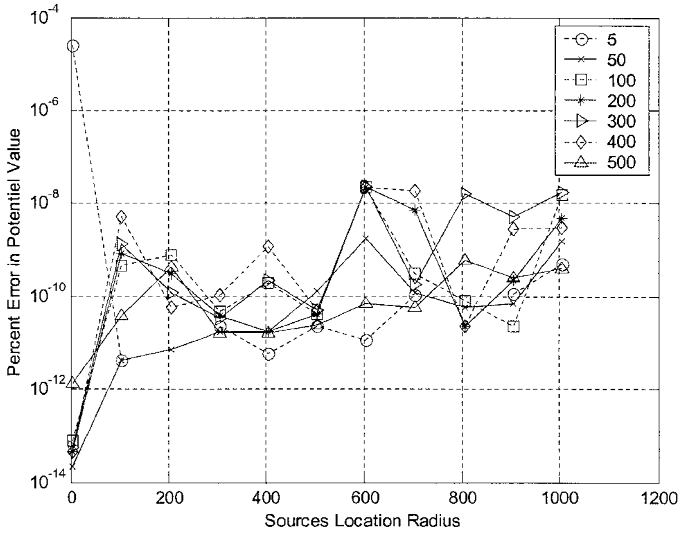

Fig.(10) – Example 1: Percent error in potential for 500 boundary points, and 5 to 500 sources.
The studied sources location radius ranging from 5 to 1005

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The effect of these variables on the condition number of the collocation matrix is also discussed. It has to be noted that the sources can be placed arbitrary, however, for simplicity in programming the preprocessing, the sources were placed on a circle (as shown in fig.(1)) which center was located at the centroid of the domain.

Example 1:

The 2×1 rectangular domain shown in fig.(2) has a constant temperature distribution along two sides, whereas the other sides have zero temperature gradient. This example had been considered by Becker [10], and has the following analytical solution:

$$u = -2x + 5, \qquad (5)$$

where the x axis is shown in fig.(2). The potential was studied at point A (fig.(2)). The errors in the results are plotted as shown in the figures (3 to 10). From the figures (3 to 10), it can be seen that:

1- In most cases, the error decreases rapidly by increasing the radius of the circle on which the sources are located, then it becomes almost constant.
2- For the same number of boundary points, when the number of sources is very small, the error is relatively high (18% in case of 12 boundary points, 2 sources at a radius of 2). By increasing the number of sources gradually, the error decreases until it reaches its minimum value (10^{-13} %).
3- Increasing the number of boundary points doesn't affect much the accuracy of the results; this is mainly due to the applied constant boundary conditions.

It can be noted that there are some discontinuities in the plotted lines in these figures: the error at these points is virtually equal to zero, which cannot be represented on a semi-log scale.

Example 2

A rectangular domain of dimensions $1\times\pi/2$ is considered (fig.(11)). Potential values of $u = \cos y$ and $u = e \cos y$ are applied on two opposite sides. The other sides have a constant potential distribution ($u = 0$) on one side and a zero flux on the other. The analytical solution, at any point inside the domain, is determined by: $u = e^x \cos y$

The potential was studied at point B as shown in fig.(11). The obtained results are shown in figures (fig.(12) to fig.(15)). This problem had been considered by Fenner [1] who placed the sources on a circle of radius range 2 to 5.

From the results, the following notes can be drawn:

1- In most cases, the error in potential increases rapidly, by increasing the radius of the circle on which the sources are placed, and then it becomes almost constant (within 0.1%).

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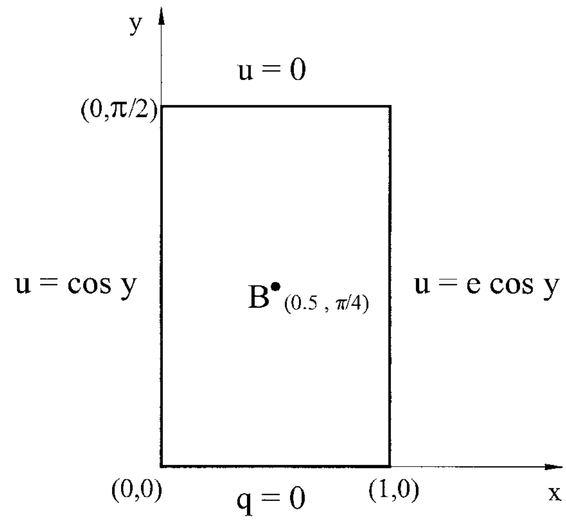

Fig.(11) – Geometry and boundary condition for example 2.

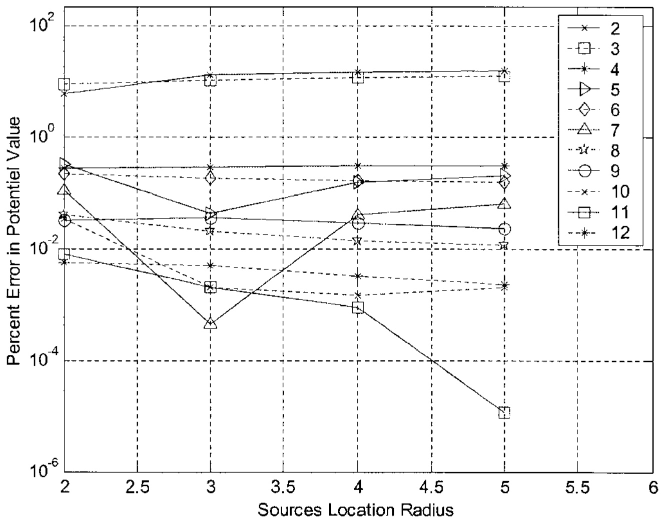

Fig.(12) – Example 2: Percent error in potential for 12 boundary points, and 2 to 12 sources.

The studied sources location radius ranging from 2 to 5.

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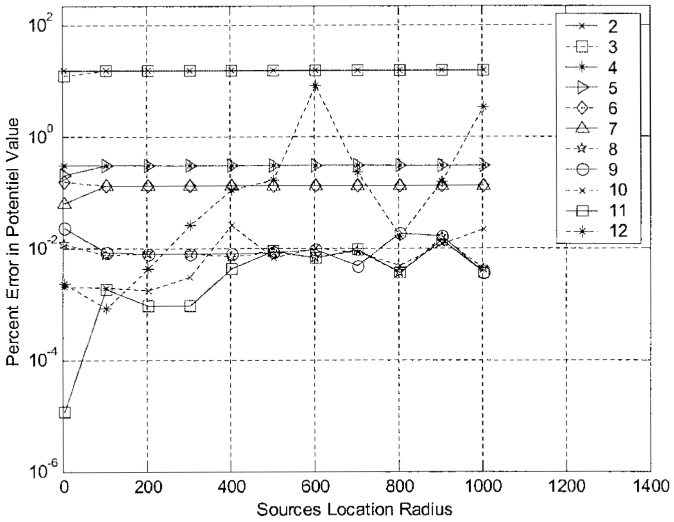

Fig.(13) – Example 2: Percent error in potential for 12 boundary points, and 2 to 12 sources.
The studied sources location radius ranging from 5 to 1005.

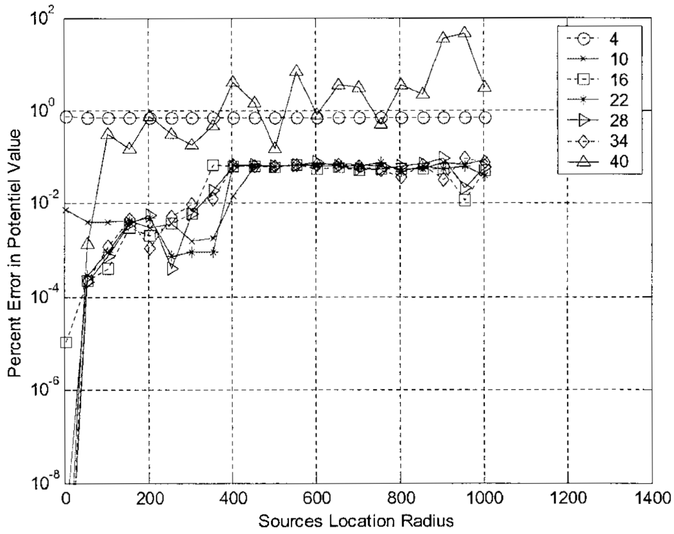

Fig.(14) – Example 2: Percent error in potential for 40 boundary points, and 4 to 40 sources.
The studied sources location radius ranging from 5 to 1005.

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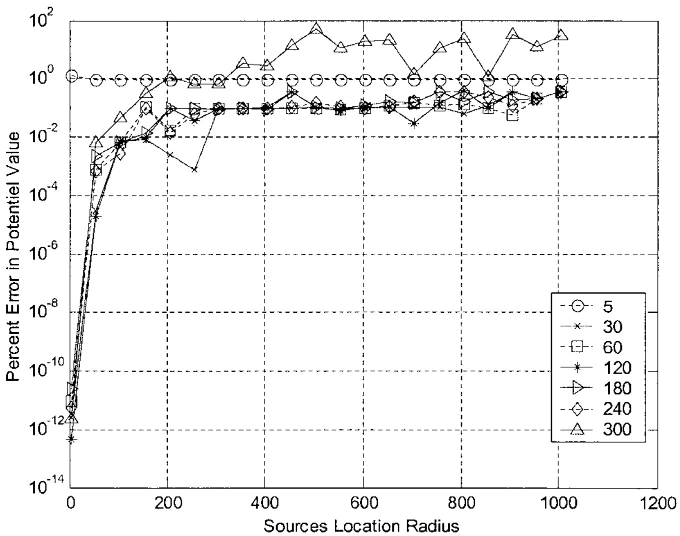

Fig.(15) – Example 2: Percent error in potential for 300 boundary points, and 5 to 300 sources.
The studied sources location radius ranging from 5 to 1005.

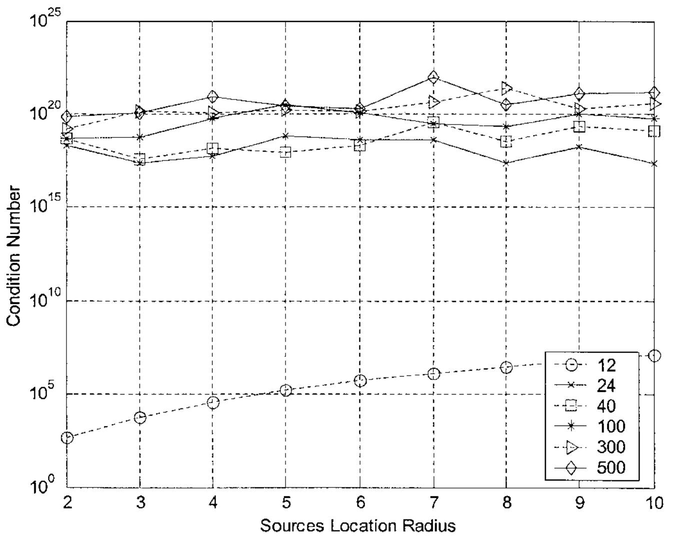

Fig.(16) – Example 1: Condition number for collocation matrix.
The studied sources location radius ranging from 2 to 10.

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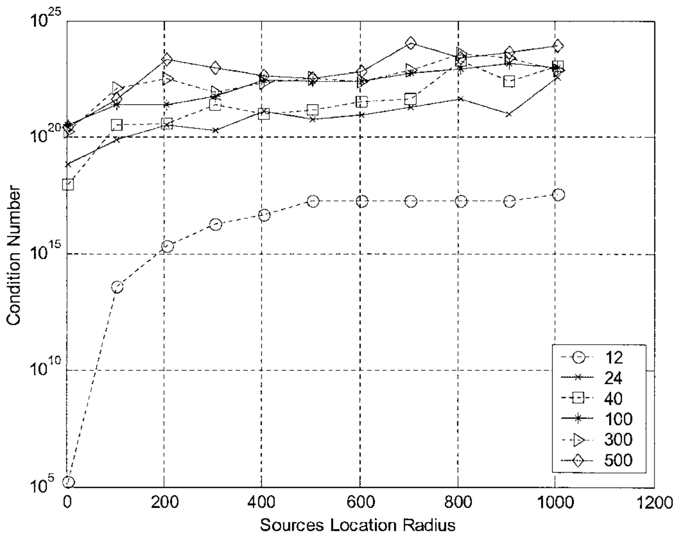

Fig.(17) – Example 1: Condition number for collocation matrix. The studied sources location radius ranging from 5 to 1005.

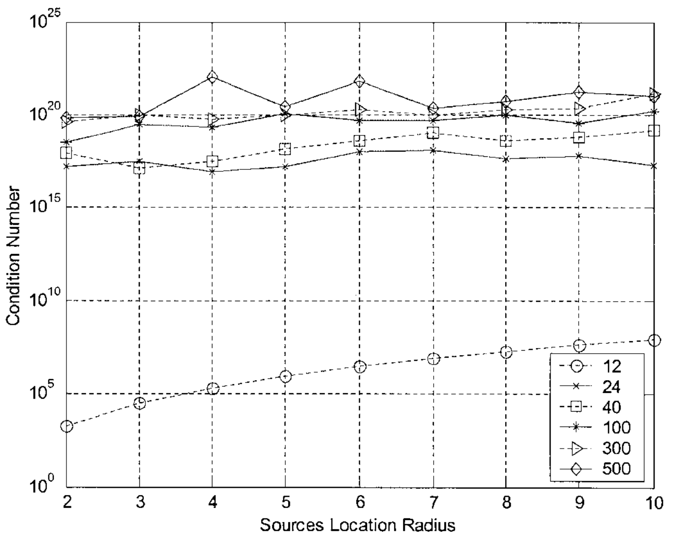

Fig.(18) – Example 2: Condition number for collocation matrix. The studied sources location radius ranging from 2 to 10.

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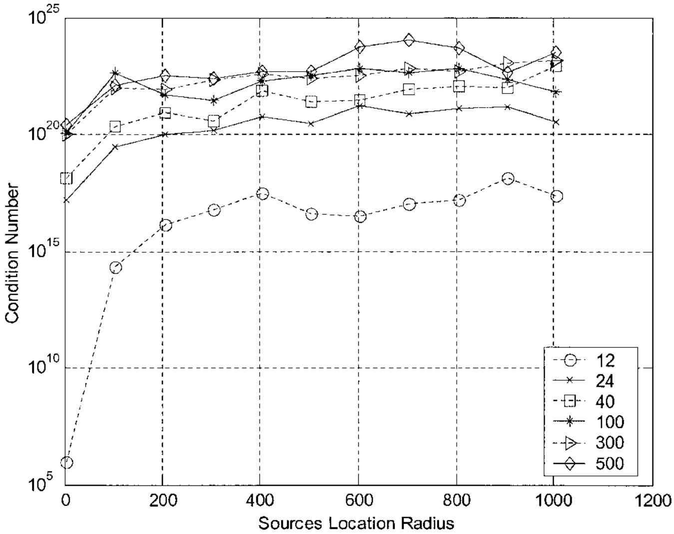

Fig.(19) – Example 2: Condition number for collocation matrix. The studied sources location radius ranging from 5 to 1005.

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2- For the same number of boundary points, the error is relatively high when the number of sources is either very small or when it is equal to the number of boundary points. The best results are obtained when the number of sources is 65-90% of the number of boundary points.

It has to be noted that Fenner [1] suggested choosing the radius between 3 and 4 to get the best results, as this range gave the best accuracy with the most stable condition number (as it will be shown later). However, this is not general, as it is shown in example (1).

Matrix Condition Number

The following figures (fig.(16 to19)) demonstrates the relationship between the condition number of the collocation matrix (equation(4)), and both the radius of the circle on which the sources are placed and the number of boundary points. The results are plotted only when the number of sources equal to the number of boundary points (The case of square collocation matrix).

It can be seen that the condition number was slightly increased by the increase of the sources location radius [11], and also increased by the increase of the number of boundary points. The best range of condition number is obtained when the number of boundary points is small and within the radius of 2 to 6; hence it seems that most of the authors in literature tried to place their sources within this limit (as Fenner did [1]). Generally, the obtained condition number is large, and this is the reason of the ill-conditioning. As mentioned in section 2, high precision software should be used to deal with such conditioning without affecting the accuracy of solutions.

4 Conclusions

In this paper, the Method of Fundamental Solutions is re-investigated using a highly accurate arithmetic software in order to be able to deal with highly ill-conditioned system of equations. The following conclusions might be drawn from the results:

1- For the sake of accuracy of the results, advanced high precision data type should be used in programming the MFS.

2- The sources can be placed arbitrary, however, from the stability point of view, it is recommended to place the sources far away from the boundary.

3- As the results obtained from the MFS are very accurate (as shown in the previous examples),there is no need to optimize the placement of sources.

4- The number of sources should be enough to allow accurate interpolation (distribution), and should not be too much as it will over interpolate the solution leading to inaccurate results. In general, it is

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recommended that the number of sources should be within 65-90% of the number of boundary points.

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6 References

1. R. T. Fenner, 'Source Field Superposition Analysis of Two Dimensional Potential Problems', International J. for Numerical Methods in Engineering, Vol. 32, pp.1079-1091, 1991.
2. Karageorghis, G. Fairweather, 'The Method of Fundamental Solutions for the Numerical Solution of Biharmonic Equation', J. of Computational Physics, Vol. 69, pp 434-459, 1987.
3. G. H. Koopmann, 'A Method for Computing Acoustic Fields Based on the Principle of Wave Superposition', L. Song, J. B. Fahnline, J. Acoust. Soc. Am., Vol. 86, No. 6, Acoustical Society of America, 1989.
4. Karageorghis, G. Fairweather, 'The Method of Fundamental Solutions for Axisymetric Elasticity Problems', Computational Mechanics, Vol. 25, pp. 524-532, Springer-Verlag, 2000.
5. G. Burgess, E. Mahajerin, 'A Comparison of The Boundary Element and Superposition Methods', Computers & structures, Vol. 19, No. 5/6, pp. 697-705, Pergamon Press Ltd., 1984.
6. Ivan Saavedra, Henry Power, 'Adaptive Refinement in Method of Fundamental Solutions for Bidimensional Laplace Problems', Boundary Element Comunications, Vol 12, No. 1, 2001.
7. P. Cisilino, B. Sensale, 'Optimal Placement for Source Points for Singular Problems in the Method of the Fundamental Solutions', Advances in Boundary Element Techniques II, Hoggar Press, 2001.
8. S. Chen, 'The Method of Fundamental Solutions for Non-linear Thermal Explosions', Communications in Numerical Methods in Engineering, Vol.11, pp. 675-681, 1995.
9. R. Hunt, R. L. Lipsman, J. M. Rosenberg, 'A Guide to Matlab for Beginners and Experienced Users', Cambridge University Pres, 2001.
10. A. Becker, 'The Boundary Element Method in Engineering, a complete course', Mc-GRAW-HILL, 1992.
11. A. Bogomolny, 'Fundamental Solution Method for Elliptic Boundary Value Problems', SIAM J. Numer. Anl. , Vol. 22, No. 4, Society for Industrial and Apply Mathematics, 1985.