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On methods for solving the Dirichlet problem for Poisson's equation

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Abstract

In this paper we present convergence rates of the dual reciprocity method (DRM) for solving the Dirichlet problem for Poisson's equation. Using a new approximation scheme recently developed in [6] we show that the method of particular solution (MPS) and method of fundamental solution (MFS) can be easily implemented in computation to provide an efficient approximation method for solving the Dirichlet problem.

1 Introduction

Recently there has been increasing interest in the application of the DRM [1] to a variety of scientific and engineering problems. The work on the DRM seems predicated on the belief that the incurred errors are the sum of the errors due to the chosen boundary element method (BEM) and the error in approximating the source term. However, there is a "see-saw" effect between the accuracy of the source term approximation and the growth of the approximate particular solutions generated by using the given basis elements. This latter property tends to increase the BEM error resulting in the possibility that using high order source term approximations can actually cause the overall algorithm to diverge [3]. This effect was also noted by Pollandt in [9] who examined the use of thin-plate spline quasi-interpolation coupled with a spline approximation method for the single layer approximation of the related Laplace's equation. His argument, though rigorous, seems to be difficult to generalize to other rbfs and to R^3. In this paper we present a way to overcome some of these difficulties by giving a general scheme to bound the errors in the DRM for Poisson's equation in R^2 and R^3 through the use of a new approximaton scheme recently developed in

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[6] . By this new approximation scheme we also show that the MPS and MFS can be easily implemented to achieve an efficient approximation of the exact solution of the Dirichlet problem avoiding the difficulty of discretizing the phyiscal domain needed in the BEM. The application of the methods discussed here is not limited to the Dirichlet problem, which we will use to examine other differential equations in future work.

The organization of this paper is as follows. In Section 2 we describe the numerical methods mentioned above. In Section 3 we present a rbf approximation scheme and the approximate particular solutions. Then we present the convergence results for the DRM and examine a numerical example by using the MPS and MFS in Section 4.

2 Description of numerical methods for solving the Dirichlet problem

2.1 MPS

The Dirichlet problem for Poisson's equation is expressed as follows:

$$\Delta u(x) = f(x), \qquad x \in \Omega, \tag{1}$$
$$u(x) = g(x), \qquad x \in \partial\Omega, \tag{2}$$

where Ω is a bounded and simply connected domain in R^s for $s = 2$ or 3, and $\partial\Omega$ is the boundary of Ω which is assumed to be smooth. (For simplicity, we assume that $\partial\Omega$ is C^∞, and f, g are C^∞ on Ω, $\partial\Omega$, as well). Theoretically, it is known that there exists a unique C^∞ function u satisfying (1)-(2) [8, for instance]. Practically, certain methods have to be used to solve (1)-(2) numerically. For instance, in the implementation of the MPS, we assume that a particular solution u_p of (1) is available, that is

$$\Delta u_p(x) = f(x), \qquad x \in \Omega, \tag{3}$$

but does not necessarily satisfy the boundary condition (2). Then, if u is the unique solution of the Dirichlet problem (1)-(2), by letting

$$v(x) = u(x) - u_p(x), \tag{4}$$

then

$$\Delta v(x) = 0, \qquad x \in \Omega, \tag{5}$$
$$v(x) = g(x) - u_p(x), \qquad x \in \partial\Omega. \tag{6}$$

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Denote by v_b the solution of the above equations (5)-(6). Then the unique solution of (1)-(2) is given by

$$u = u_p + v_b. \tag{7}$$

The BEM can be used to solve (5)-(6), which we review below for the double layer potential only.

2.2 BEM via double layer potential

It is known that the exact solution v of (5)-(6) can be expressed by the following double layer potential [2]

$$v(x) = \int_{\partial\Omega} \frac{\partial}{\partial n_y} G(x,y)\sigma(y)ds, \qquad x \in \Omega, \tag{8}$$

$$\sigma(x) = -2\int_{\partial\Omega} \frac{\partial}{\partial n_y} G(x,y)\sigma(y)ds - 2(g(x) - u_p(x)), \qquad x \in \partial\Omega, \tag{9}$$

where $\frac{\partial}{\partial n_y}$ denotes the normal derivative, and $G(x,y)$ is the fundamental solution of Laplace's equation, given by

$$G(x,y) = \begin{cases} \frac{1}{2\pi}\ln\|x-y\|, & in\ R^2, \\ \frac{-1}{4\pi\|x-y\|}, & in\ R^3. \end{cases} \tag{10}$$

In general, σ in (9) can not be computed explicitly. For practical purposes, one obtains an approximate solution σ_m of (9) by some numerical method such as a collocation or a Galerkin method [2, for details], which results in an approximate solution v_m of (8). Below is a convergence result for v_m.

Theorem 1 *Let v_m be the numerical approximation to the exact solution v of (8)-(9) by the double layer potential, then*

$$\|v - v_m\|_{\mathcal{L}^2(\Omega)} \le cM(m)^{-\lambda} \left\|(g-u_p)^{(l)}\right\|_{\mathcal{L}^\infty(\partial\Omega)}, \tag{11}$$

where $M(m)$ is the dimension of the solution space, and λ and l depend on the smoothness of the solution and the method chosen for finding the numerical solution.

2.3 MFS

Another method for solving the homogeneous equation (8)-(9) is the MFS, in which we choose a fictitious domain $\tilde{\Omega}$ such that $\Omega \subset \tilde{\Omega}$, and assume the

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approximate solution v_m to the exact solution of (5)-(6) can expressed as a linear combination of fundamental solutions

$$v_m(x) = \sum_{i=1}^{m} c_i G(x, y_i), \qquad x \in \Omega, \tag{12}$$

where $G(x, y)$ is the fundamental solution given in (10), and the source points, or singularities, $\{y_i\}_1^m$ are placed on the boundary of $\tilde{\Omega}$. Notice that v_m automatically satisfies the differential equation (5). All we need to do is to enforce v_m so that it satisfies the boundary condition (6). By the collocation method, we choose the same number of collocation points as the number of source points on the physical boundary of Ω. Let $\{x_i\}_1^m \in \partial\Omega$, then set

$$\sum_{i=1}^{m} c_i G(x_j, y_i) = g(x_j) - u_p(x_j), \qquad 1 \leq j \leq m, \tag{13}$$

which can be used to solve for $\{c_i\}_1^m$. Theoretical results on the convergence of the MFS have been derived in [4] and several other papers. For instance, if $\partial\Omega$ is a chosen Jordan curve in the plane and the data are analytic, then

$$\|v - v_m\|_\infty \leq c(r/R)^m, \tag{14}$$

where r and R are the diameters of Ω and $\tilde{\Omega}$, respectively.

2.4 Approximate solution of the Dirichlet problem

Numerically, to find a particular solution of (1), the following technique has found widespread use in the engineering literature. Begin by approximating f by a linear combination of basis functions $\{\phi_j(x)\}_{j=1}^N$. That is,

$$f \approx \hat{f} = \sum_{j=1}^{N} \alpha_j \phi_j, \tag{15}$$

where the coefficients are determined in a manner so that $\hat{f}$ is a good approximation to f. Then one defines

$$u_N = \sum_{j=1}^{N} \alpha_j \psi_j \tag{16}$$

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where ψ_j satisfies

$$\Delta\psi_j = \phi_j, \qquad 1 \le j \le N. \tag{17}$$

By linearity, it follows immediately that

$$\Delta u_N = \hat{f} \tag{18}$$

so that u_N can be thought of as an "approximate" particular solution. Then rather than solving (8)-(9) for v with respect to $g(x) - u_p(x)$ in (9), we solve (8)-(9) with respect to $g(x) - u_N(x)$ by either the BEM or the MFS to get an approximate solution $v_{N,m}$, and

$$\hat{u} = v_{N,m} + u_N \tag{19}$$

is considered as an approximation to the solution u of (1)-(2).

3 Approximate particular solutions

3.1 Rbf approximation

We now describe an approximation scheme recently developed in Li and Micchelli [6]. First we introduce some notation. For $\delta > 0$, let $\Omega_\delta = \Omega + \delta I$, where $I = [-1,1]^s$. For any integer n, set

$$I_n(\Omega_\delta) = \left\{ \mathbf{j} \in Z^s : \left[\frac{\mathbf{j}}{n}, \frac{\mathbf{j}+\mathbf{1}}{n}\right]^s \cap \Omega_\delta \neq 0 \right\}, \tag{20}$$

where $\mathbf{1} := (1, \cdots, 1) \in Z^s$. Denote by $\mathcal{W}^{p,1}(\Omega)$ the space of all functions f whose gradient is in $\mathcal{L}^p(\Omega)$ with the usual Sobolev norm

$$\|f\|_{\mathcal{W}^{p,1}(\Omega)} = \sum_{k=1}^{s} \|\frac{\partial f}{\partial x_k}\|_{\mathcal{L}^p(\Omega)} + \|f\|_{\mathcal{L}^p(\Omega)}. \tag{21}$$

For a function $f \in \mathcal{W}^{p,1}(\Omega_\delta)$, one can choose a function χ of a certain degree of smoothness such that χ is identical to 1 on the closure $\overline{\Omega}$ and vanishes outside of Ω_δ. Let $f_\chi = f \cdot \chi$, then $f_\chi \in \mathcal{W}^{p,1}(R^s)$ and it is compactly supported in $\overline{\Omega_\delta}$. Denote by $\mathcal{W}_0^{p,1}(\Omega_\delta)$ the subspace of functions in $\mathcal{W}^{p,1}(R^s)$ which vanishes outside of Ω_δ. We then consider the approximation of functions in $\mathcal{W}_0^{p,1}(\Omega_\delta)$ over the region Ω. Suppose that $\phi \in \mathcal{L}^1(R^s)$ is given with the property that

$$\int_{R^s} \phi(\mathbf{x})d\mathbf{x} = 1. \tag{22}$$

For every $f \in \mathcal{W}_0^{p,1}(\Omega_\delta)$ and an integer $n \in N$, let

$$\mathcal{B}_n f = \frac{1}{n^{s(1-\gamma)}} \sum_{\mathbf{j} \in I_n(\Omega_\delta)} f(\frac{\mathbf{j}}{n})\phi(n^\gamma \cdot -\mathbf{j}n^{\gamma-1}). \tag{23}$$

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Denote by $\mathcal{S}^\beta(R^s)$ the set of all functions f satisfying

$$|f(\mathbf{x})| \leq c(1+|\mathbf{x}|)^{-\beta}, \tag{24}$$

for all $\mathbf{x} = (x_1, \cdots, x_s) \in R^s$, where $|\mathbf{x}| := \max\{|x_i|; 1 \leq i \leq s\}$ and c is a positive constant. The following result is shown in Li and Micchelli [6].

Theorem 2 *Suppose that $\phi \in \mathcal{W}^{p,1}(R^s) \cap \mathcal{S}^\alpha(R^s)$ for some $\alpha > s$ and $0 < \lambda < \gamma < \frac{1}{1+s/q}$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then there exists a positive constant c such that for all $f \in \mathcal{W}_0^{p,1}(\Omega_\delta)$ and large $n \in N$ there holds the inequality*

$$\|\mathcal{B}_n f - f\|_{\mathcal{L}^p(\Omega)} \leq \frac{c}{n^\eta} \|f\|_{\mathcal{W}^{p,1}(\Omega_\delta)} \tag{25}$$

where $\eta := \min\{\lambda, (\gamma-\lambda)(\alpha - s), 1 - \gamma - s\gamma/q\}$.

Below we give an example to illustrate the approximatoin error.

Example 1

Consider $f(x,y) = 2e^{(x-y)}$ in R^2. Let

$$T(t) = \begin{cases} 0, & t \leq 0, \\ \frac{9t^3}{2\delta^3}, & 0 \leq t \leq \frac{\delta}{3}, \\ \frac{1}{2} - \frac{9t^3}{\delta^3} + \frac{27t^2}{2\delta^2} - \frac{9t}{2\delta}, & \frac{\delta}{3} \leq t \leq \frac{2\delta}{3}, \\ \frac{-7}{2} + \frac{9t^3}{2\delta^3} - \frac{27t^2}{2\delta^2} + \frac{27t}{2\delta}, & \frac{2\delta}{3} \leq t \leq \delta, \\ 1, & t \geq \delta, \end{cases} \tag{26}$$

which is a piecewise cubic polynomial in C^2, and let

$$W(t) = T(\delta + h + t)T(1 + \delta + h - t). \tag{27}$$

Then W is compactly supported in $[-\delta - h, 1 + \delta + h]$, and it takes on 1 when $t \in [-\delta, 1+\delta]$. Figure 1 is the graph of W when $\delta = 0.05$ and $h = 0.5$.

We then set

$$f_X(x,y) = f(x,y)W(x)W(y) \tag{28}$$

and consider the approximation on the domain $D = [0,1]^2$. The basis function that we use in (23) is

$$\phi(r) = \frac{3}{\pi(r^2+1)^4}, \qquad r = \sqrt{x^2+y^2}, \tag{29}$$

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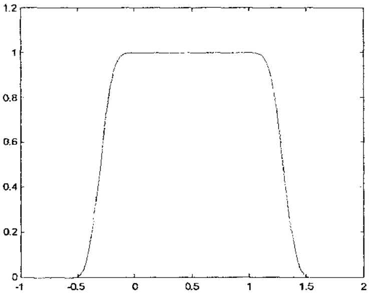

Figure 1: The graph of W

which is an inverse multiquadric. Below is a table of the absolute approximation error $\mathcal{B}_n f - f$ at certain specified points with respect to different choices of n, where we set $\delta = h = 0.05$.

	n=20	50	100	150
(0.5000, 0.5000)	7.44e-003	1.52e-003	5.07e-004	2.73e-004
(0.3824, 0.1251)	2.20e-004	1.72e-003	6.35e-004	3.50e-004
(0.2672, 0.8132)	2.46e-003	7.73e-004	2.88e-004	1.58e-004
(0.9013, 0.0560)	5.77e-002	1.53e-004	9.78e-004	6.01e-004

3.2 Bounds on the norms of approximate particular solutions by rbfs

To use rbfs to approximate the source term f, it is convenient to study Poisson's equation expressed in radial form as follows:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi(r)}{dr}\right) = \phi(r) \tag{30}$$

in R^2 and

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi(r)}{dr}\right) = \phi(r) \tag{31}$$

in R^3. By straightforward integration, $\psi(r)$ is given by the integrals

$$\psi(r) = -r^2\int_0^1 s\ln s\phi(sr)ds + A\ln r + B \tag{32}$$

in R^2 and

$$\psi(r) = r^2\int_0^1 s(1-s)\phi(sr)ds + \frac{A}{r} + B \tag{33}$$

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in R^3. Since we want $\psi(r)$ to be continuous at $r = 0$ we set $A = B = 0$ in both (32) and (33). As shown for example in [2], $\psi(r)$ can be computed explicitly for many of the standard rbfs such as thin plate splines, multiquadrics and compactly supported rbfs. For Gaussians, $\psi(r)$ can be determined in terms of exponential integral functions.

To derive the convergence results for the DRM, we need the estimates on the norms of approximate particular solutions. Let $\psi(r)$ satisfy $\Delta\psi(r) = \phi(r)$. Then an approximate particular solution corresponding to (23) is given by

$$u_n(x) = \frac{1}{n^{s(1-\tau)}} \sum_{j \in I_n(\Omega_\delta)} f\left(\frac{j}{n}\right) n^{-2\tau} \psi(n^\tau(x - j/n)) \qquad (34)$$

for $s = 2$ or 3.

For the sake of discussion, we consider radial basis functions of the form $\Phi(r^2)$ with the property that

$$\int_{R^s} \Phi(r^2)dx = 1, \qquad (35)$$

and moreover we require that Φ is l times continuously differentible and its derivatives decay in the following order

$$\frac{d^i\Phi}{dx^i}(x^2) = O(x^{-i}), \qquad x \to \infty, \qquad (36)$$

for $0 \leq i \leq l$. Most of the commonly used rbfs including Gaussians, compactly supported rbfs, multiquadrics and inverse multiquadrics can be made to satisfy (35)-(36) [7]. The following results on the norms of approximate particular solutions hold.

Proposition 3 *Suppose that Φ satisfies (35) and (36), and $\Delta\Psi = \Phi(r^2)$. Let u_n be the approximate particular solution given in (34). Then*

$$\|u_n\|_{\mathcal{W}^{l,p}(\partial\Omega)} \leq cn^{(l+s)\tau}, \qquad (37)$$

for any $p, 1 \leq p \leq \infty$, where $s = 2$ or 3.

Similar results on thin plate splines are also be derived [5].

4 Theorectical and numerical results

Let u_n be the approximate particular solution of (1) given in (27). Denote by $v_{n,m}$ the numerical solution of

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$$\begin{aligned} \Delta v(x) &= 0, \qquad x \in \Omega, &(38)\\ v(x) &= g(x) - u_n(x), \qquad x \in \partial\Omega, &(39)\end{aligned}$$

obtained by solving the double layer potential equation. Then we consider

$$u_{n,m} = u_n + v_{n,m} \tag{40}$$

as a numerical (or approximate) solution of (1)-(2). By applying Theorems 1 and 2, and Proposition 3, the following convergence result can be derived [5, for details].

Theorem 4 *Let u be the exact solution of (1)-(2). Let $\Phi(r^2)$ satisfy (35)-(36), and $u_{n,m}$ be a numerical solution of (1)-(2) given by (40), then*

$$\|u_{n,m} - u\|_{\mathcal{L}^2(\Omega)} \leq \frac{c}{n^\sigma} \|f\|_{\mathcal{H}^1(\Omega_\delta)} + cM(m)^{-\lambda} n^{\tau(l+s)}, \tag{41}$$

where $M(m)^{-\lambda}$ depends on the method used for solving the double layer potential equation.

Below we give an exmaple of solving the Dirichlet problem in R^2 by using the MPS and MFS.

Example 2

Consider

$$\begin{aligned} \Delta u(x,y) &= 2e^{(x-y)}, \qquad (x,y) \in \Omega, &(42)\\ u(x,y) &= e^{(x-y)}, \qquad (x,y) \in \partial\Omega. &(43)\end{aligned}$$

Assume that Ω is the domain given by $(x-1/2)^2 + (y-1/2)^2 \leq (1/2)^2$. Let $\phi(r) = \frac{3}{\pi(r^2+1)^4}$, as in Example 1, and $\Delta\psi(r) = \phi(r)$, then

$$\psi(r) = \frac{1}{4\pi}\ln\left(r^2+1\right) - \frac{1}{4\pi\left(r^2+1\right)} - \frac{1}{8\pi\left(r^2+1\right)^2} + \frac{3}{8\pi}. \tag{44}$$

Let u_n be the approximate particular solution given in (23) with respect to ψ, and we then use MFS to solve the following homogeneous equation

$$\begin{aligned} \Delta v(x,y) &= 0, \qquad (x,y) \in \Omega, &(45)\\ u(x,y) &= e^{(x-y)} - u_n(x,y), \qquad (x,y) \in \partial\Omega. &(46)\end{aligned}$$

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Choose the fictitious domain $\tilde{\Omega}$ to be $(x - 1/2)^2 + (y - 1/2)^2 \leq (3)^2$. Let the source and collocation points be equally distributed on $\partial\Omega$ and $\partial\tilde{\Omega}$, respectively, where we use m for the number of points. Let $v_{m,n}$ be the approximate solution of (45)-(46) by the MFS. Then we consider $u_{m,n} = u_n + v_{m,n}$ as an approximate solution of (42)-(43). The exact solution of (42)-(43) is $u_{exact} = e^{(x-y)}$. Below is a table of the absolute approximation error of $u_{m,n} - u_{exact}$ at certain tested points inside Ω, where we choose $n = 100$ in the approximation of $2e^{(x-y)}$, and use different values of m. It seems that $m = 10$ already gives an efficient approximation result.

	m=5	10	15	20
(0.6000,0.7312)	2.06e-003	4.29e-006	2.19e-005	2.24e-005
(0.4319,0.3217)	4.86e-003	3.82e-005	2.87e-005	2.89e-005
(0.5000,0.9814)	6.39e-003	7.44e-004	3.69e-005	1.81e-006
(0.1098,0.5126)	5.87e-003	2.43e-005	2.34e-005	1.02e-005

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