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The analog equation method for large deflection analysis of heterogeneous anisotropic membranes: a boundary-only solution

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Abstract

The Analog Equation Method (AEM) is applied to non-linear analysis of heterogeneous anisotropic membranes with arbitrary shape. In this case, the response of the membrane is described by three coupled non-linear differential equations with variable coefficients. The present formulation, being in terms of the three displacements components, permits the application of geometrical in-plane boundary conditions. The membrane is prestressed either by prescribed boundary displacements or by tractions. Using the concept of the analog equation, the three coupled non-linear equations are replaced by three uncoupled Poisson's equations with fictitious sources under the same boundary conditions. The fictitious sources are established using a procedure based on BEM and the displacement components as well as the stress resultants are evaluated from their integral representations at any point of the membrane. Several membranes are analyzed which illustrate the method and demonstrate its capabilities. Moreover, useful conclusions are drawn for the non-linear response of heterogeneous anisotropic membranes. The method has all the advantages of the pure BEM, since the discretization and integration are limited only to the boundary.

1 Introduction

Structural membranes are usually made from composite materials, which exhibit orthogonal anisotropy. Heterogeneity may also appear. For example membranes

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of variable thickness are used, when it is required, to reinforce the membrane locally or to control its transverse displacements.

In the linear membrane theory, we assume that the additional stretching of the membrane due to the in-service transverse load is small and the stress resultants are predetermined and remain unchanged during the out-of-plane deformation. However, with increasing the transverse load, the additional stretching of the membrane can not be neglected. A consequence of this is that the resulting differential equations governing the equilibrium of the membrane are coupled and non-linear. In heterogeneous anisotropic membranes the governing equations are even more complicated as their coefficients are variable.

The solution of the membrane differential equations is a very difficult mathematical problem. Analytical solutions are available only for homogeneous isotropic membranes and for simple geometry, such as circular membranes under axisymmetric loading where the problem becomes one-dimensional. Approximate and numerical solutions are also available in the literature (see [1]). The FEM has been employed for large deflection analysis of homogeneous, both isotropic and anisotropic, membranes. However, to the author's knowledge, no FEM solutions are available for heterogeneous anisotropic membranes. The BEM has been also employed for large deflection analysis of homogeneous isotropic membranes. Katsikadelis and Nerantzaki [2] developed a D/BEM approach for the solution of this problem, which was further developed by Katsikadelis et al. [1] to boundary-only BEM.

In this paper, the equations for large deflections analysis of heterogeneous anisotropic membranes are derived and a boundary-only solution is developed to solve the resulting three coupled non-linear differential equations with variable coefficients. Without restricting the generality, orthogonally anisotropic membranes are treated since this is the usual case in engineering practice. The method is based on the concept of the analog equation, according to which the three coupled non-linear differential equations governing the equilibrium of the membrane are replaced by three Poisson's equations with fictitious domain source under the same boundary conditions. The fictitious sources are established using a procedure based on BEM as it was developed for non-linear problems [3]. The three displacement components as well as their derivatives are computed from their integral representations, which are used as mathematical formulas. The stress resultants at any interior point and the reactions at any boundary point are also evaluated. Several membranes are analyzed to illustrate the applicability, efficiency and accuracy of the method. Moreover, useful conclusions are drawn concerning the response of the heterogeneous anisotropic membranes.

2 Problem statement and governing equations

Consider a thin flexible initially flat elastic membrane consisting of heterogeneous anisotropic linearly elastic material occupying the two-dimensional, in general multiply connected, domain Ω in xy-plane bounded by

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the $K+1$ nonintersecting contours $\Gamma_0, \Gamma_1, \ldots \Gamma_K$ .The membrane is prestressed either by imposed displacement $\tilde{u}, \tilde{v}$ or by tractions $\tilde{T}_x, \tilde{T}_y$ acting along the boundary $\Gamma = \cup_{i=0}^{i=K} \Gamma_i$. Moderate large deflections are considered. They result from non-linear kinematic relations, which retain the square of the slopes of the deflection surface. Thus, the strain components are given as

$$\varepsilon_x = u,_x + \frac{1}{2} w,_x^2, \quad \varepsilon_y = v,_y + \frac{1}{2} w,_y^2, \quad \gamma_{xy} = u,_y + v,_x + w,_x w,_y \quad \text{(1a,b,c)}$$

where $u = u(x,y)$, $v = v(x,y)$ are the in-plane displacement component and $w = w(x,y)$ the transverse deflection produced when the membrane is subjected to the lateral load $g = g(x,y)$ acting in the direction normal to its plane.

For orthotropic linearly elastic material the stress strain relations are given as [4]

$$\sigma_x = \frac{E_1}{1-\nu_1\nu_2}(\varepsilon_x + \nu_2\varepsilon_y), \quad \sigma_y = \frac{E_2}{1-\nu_1\nu_2}(\varepsilon_y + \nu_1\varepsilon_x), \quad \tau_{xy} = G\gamma_{xy} \quad \text{(2a,b,c)}$$

where E_1, E_2 and ν_1, ν_2 ($E_1\nu_2 = E_2\nu_1$) are the elastic moduli and the Poisson coefficients in the x and y directions, respectively; and G is the shear modulus.

The total potential of the deformed membrane is given as

$$\begin{aligned}\Pi = &\frac{1}{2}\int_\Omega (C_1\varepsilon_x^2 + C_2\varepsilon_y^2 + 2C\varepsilon_x\varepsilon_y + C_{12}\gamma_{xy}^2)d\Omega \\ &-\int_\Omega gw d\Omega - \int_\Gamma (\tilde{T}_x u + \tilde{T}_y v + \tilde{V} w)ds\end{aligned} \quad (3)$$

where $\tilde{V}$ is the prescribed transverse force per unit length along the boundary. The coefficients $C_1 = C_1(x,y)$, $C_2 = C_2(x,y)$, $C = C(x,y)$ and $C_{12} = C_{12}(x,y)$, which characterize the membrane stiffness are given as

$$C_1 = \frac{E_1 h}{1-\nu_1\nu_2}, \qquad C_2 = \frac{E_2 h}{1-\nu_1\nu_2}, \quad \text{(4a,b)}$$

$$C = \frac{E_1\nu_2 h}{1-\nu_1\nu_2} = \frac{E_2\nu_1 h}{1-\nu_1\nu_2}, \qquad C_{12} = Gh \quad \text{(4c,d)}$$

The vanishing of the first variation of the total potential, $\delta\Pi = 0$, yields the following differential equations, which govern the equilibrium of heterogeneous orthogonally anisotropic membrane in terms of the displacement components

$$(C_1 u,_x + Cv,_y),_x + (C_{12}u,_y + C_{12}v,_x),_y = -(\frac{C_1}{2} w,_x^2 + \frac{C}{2} w,_y^2),_x - (C_{12} w,_x w,_y),_y \quad \text{(5a)}$$

$$(C_2 v,_y + Cu,_x),_y + (C_{12}u,_y + C_{12}v,_x),_x = -(\frac{C_2}{2} w,_y^2 + \frac{C}{2} w,_x^2),_y - (C_{12} w,_x w,_y),_x \quad \text{(5b)}$$

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$$[C_1(u,_x+\frac{1}{2}w,_x^2)+C(v,_y+\frac{1}{2}w,_y^2)]w,_{xx}+2[C_{12}(u,_y+v,_x+w,_x w,_y)]w,_{xy}+ \\ [C_2(v,_y+\frac{1}{2}w,_y^2)+C(u,_x+\frac{1}{2}w,_x^2)]w,_{yy}=-g \tag{5c}$$

in Ω together with the boundary conditions on Γ

$$T_x=\tilde{T}_x \quad \text{or} \quad u=\tilde{u} \tag{6a}$$

$$T_y=\tilde{T}_y \quad \text{or} \quad v=\tilde{v} \tag{6b}$$

$$T_x w,_x+T_y w,_y=\tilde{V} \quad \text{or} \quad w=\tilde{w} \tag{6c}$$

where

$$T_x=N_x\cos a+N_{xy}\sin a\,, \qquad T_y=N_{xy}\cos a+N_y\sin a \tag{7a,b}$$

are the components of the boundary tractions; $a=\measuredangle x,\mathbf{n}$. The tilde over a symbol designates prescribed quantity. It should be noted that mixed boundary conditions could also be applied. N_x, N_y, N_{xy} are the membrane forces in Ω given as

$$N_x=\sigma_x h\,, \qquad N_y=\sigma_y h\,, \qquad N_{xy}=\tau_{xy}h \tag{8a,b,c}$$

The prestress can be applied either before the action of the transverse load or simultaneously. In the first case, the transverse load should be applied with homogeneous in-plane boundary conditions, i.e. $u=0$, $v=0$ and the terms in brackets (membrane forces) in eqn (5c) should be augmented by those resulting from the prestress. Since T_x and T_y depend on the squares of $w,_x$ and $w,_y$, we can readily conclude that the boundary conditions are non-linear, when the tractions are prescribed. In this analysis, without restricting the generality, it is assumed that the membrane is prestressed by imposed boundary displacements acting simultaneously with the transverse load. Namely, the assumed boundary conditions are

$$u=\tilde{u}\,, \qquad v=\tilde{v}\,, \qquad w=\tilde{w} \tag{9a,b,c}$$

When the membrane is prestressed by boundary tractions, the displacements $\tilde{u},\tilde{v}$ are first established by solving a plane stress problem. In any case, attention should be paid, so that the prestress will result in tensile forces N_1, N_2 in the principal directions to avoid wrinkling of the membrane, namely

$$N_{1,2}=\frac{N_x+N_y}{2}\pm\sqrt{(\frac{N_x-N_y}{2})^2+N_{xy}^2}\;\rangle 0 \tag{10}$$

3 The analog equation method for large deflections of membranes

The boundary value problem described by eqns (5) and (9) is solved using the Analog Equation Method (AEM). Detailed description of the AEM for large

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deflection analysis of membranes is presented in [1]. However, for the completeness of this paper the method is concisely prescribed here.

According to the concept of the analog equation [3], eqns (5) are replaced with three Poisson's equations

$$\nabla^2 u_i = b_i \quad (i = 1,\, 2,\, 3) \tag{11}$$

where $b_i = b_i(x_1, x_2)$ are fictitious sources. Note that here $u_1,\, u_2, u_3$ stand for the functions u, v, w, respectively. The fictitious sources are established using BEM. For this purpose b_i is approximated by

$$b_i = \sum_{j=1}^{M} a_j^{(i)} f_j \tag{12}$$

where f_j are approximation radial base functions and $a_j^{(i)}$ are $3M$ coefficients to be determined. We look for a solution in the form $\overline{u}_i + u_i^p$ where $\overline{u}_i$ is the homogeneous solution and u_i^p a particular one. The particular solution is obtained as

$$u_i^p = \sum a_j^{(i)} \hat{u}_j \tag{13}$$

where $\hat{u}_j$ is a particular solution of

$$\nabla^2 \hat{u}_j = f_j \tag{14}$$

The homogenous solution is obtained from the boundary value problem

$$\nabla^2 \overline{u}_i = 0 \quad \text{in } \Omega \quad \text{and} \quad \overline{u}_i = \tilde{u}_i - \sum_{j=1}^{M} a_j^{(i)} \hat{u}_j \quad \text{on } \Gamma \tag{15a,b}$$

Thus, writing the solution of the homogeneous eqn (15a) in integral form, the solution of eqn (11) is given as

$$cu_i = -\int_\Gamma (u^* \overline{u}_{i,n} - \overline{u}_i u^*_{,n})ds + \sum_{j=1}^{M} a_j^{(i)} \hat{u}_j \qquad i = 1,2,3 \tag{16}$$

with $u^* = \ell nr/2\pi$, $r = |P - Q|$, $Q \in \Gamma$ being the fundamental solution of the Laplace equation and $c = 1, 1/2, 0$ depending on whether $P \in \Omega$, $P \in \Gamma$, $P \notin \Omega \cup \Gamma$, respectively. The first and second derivatives of the displacements for points inside Ω $(c = 1)$ are obtained by direct differentiation of eqn (16). Thus, we have

$$u_{i,k} = -\int u^*_{,k} \overline{u}_{i,n} - \overline{u}_i u^*_{,nk})ds + \sum a_j^{(i)} \hat{u}_{j,k} \qquad (k = 1,2) \tag{17}$$

$$u_{i,kl} = -\int (u^*_{,kl} \overline{u}_{i,n} - \overline{u}_i u^*_{,nkl})ds + \sum a_j^{(i)} u_{j,k\ell} \qquad (k,l = 1,2) \tag{18}$$

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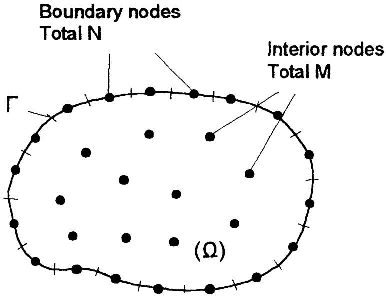

Figure 1: Boundary discretization and domain nodal points

Using the BEM with N constant boundary elements, discretizing eqn (16) and applying it to the N boundary nodal points yield

$$\tfrac{1}{2}\tilde{\mathbf{u}}_i = \mathbf{H}\bar{\mathbf{u}}_i - \mathbf{G}\bar{\mathbf{u}}_{i,n} + \hat{\mathbf{U}}\mathbf{a}^{(i)} \qquad (i = 1, 2, 3) \tag{19}$$

where $\mathbf{H}, \mathbf{G}$ are $N \times N$ and $\hat{\mathbf{U}}$ $N \times M$ known matrices and $\mathbf{a}^{(i)}$ is the vector of the unknown coefficients. Eqns (16), (17) and (18) are subsequently applied to M points inside the domain $(c = 1)$ (see Fig. 1). This yields after elimination of $\bar{\mathbf{u}}$ and $\bar{\mathbf{u}}_n$ by virtue of the boundary conditions (15b).

$$\mathbf{u}_i = \mathbf{D}\mathbf{a}^{(i)} + \mathbf{E}\tilde{\mathbf{u}}_i \tag{20}$$

$$\mathbf{u}_{i,k} = \mathbf{D}_k\mathbf{a}^{(i)} + \mathbf{E}_k\tilde{\mathbf{u}}_i \tag{21}$$

$$\mathbf{u}_{i,kl} = \mathbf{D}_{kl}\mathbf{a}^{(i)} + \mathbf{E}_{kl}\tilde{\mathbf{u}}_i \tag{22}$$

where ($i = 1, 2, 3$; $k, l = 1, 2$) and $\mathbf{D}, \mathbf{E}, \ldots, \mathbf{E}_{kl}$ are known matrices.

The final step of AEM is to apply eqns (5) to the M points inside Ω and substitute u_i and their derivatives from eqns (20)–(22). This yields

$$\mathbf{a}^{(1)} = \mathbf{F}_1(\mathbf{a}^{(3)}) \tag{23a}$$

$$\mathbf{a}^{(2)} = \mathbf{F}_2(\mathbf{a}^{(3)}) \tag{23b}$$

$$\mathbf{a}^{(3)} = \mathbf{F}_3(\mathbf{a}^{(3)}) \tag{23c}$$

Eqns (23) can be solved numerically to evaluate $\{a^{(i)}\}$. Note that eqn (23c) is non-linear and it can be solved using any method for solving non-linear algebraic equations. In this investigation the fixed point method has been employed after modifying $\mathbf{F}_3(\mathbf{a}^{(3)})$ appropriately, so that it becomes a contraction mapping. This guarantees the convergence, particularly in the case of zero prestress, where the other methods (e.g. Newton-Raphson) fail to converge. Once the coefficients $\alpha_j^{(i)}$ have been evaluated the displacements and their derivatives at the M points are computed from eqns (20)-(22). For points $P \in \Omega$ not coinciding with

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the nodal points these quantities are evaluated from the discretized counterparts of eqns (16)–(18).

4 Numerical Examples

On the basis of the numerical procedure presented in the previous section a FORTRAN code has been written and numerical results for certain membranes have been obtained, which illustrate the effectiveness of the method. The employed approximation functions f_j are the multiquadrics [5] which are defined as

$$f_j = \sqrt{r^2 + c^2} \tag{24}$$

where c is an arbitrary constant and

$$r = \sqrt{(x - x_j)^2 + (y - y_j)^2} \qquad j = 1, 2, \ldots, M \tag{25}$$

with x_j, y_j being the collocation point. Using these radial base functions the particular solution of eqn (14) is obtained as

$$\hat{u}_j = -\frac{c^3}{3} \ell n(c\sqrt{r^2 + c^2} + c^2) + \frac{1}{9}(r^2 + 4c^2)\sqrt{r^2 + c^2} \tag{26}$$

The numerical results are presented using the following non-dimensional quantities.

$$\bar{u} = u/a\,,\ \bar{v} = v/a,\ \bar{w} = w/a\,,\ \bar{N} = N\frac{(1-\nu_1^2)}{Eh},\ \bar{g} = g\frac{a(1-\nu_1^2)}{Eh} \tag{27}$$

where a denotes a characteristic length of the membrane and N a sectional force per unit length.

4.1 Square anisotropic membrane

A square membrane with side length $a = 5.0m$ has been analyzed ($N = 100$, $M = 49$). The employed data are $E_1 = E/\sqrt{\lambda}$, $E_2 = E\sqrt{\lambda}$; $\nu_1 = 0.3$, $\nu_2 = \lambda\nu_1$. First the membrane was analyzed with zero prestress and $\lambda = 1.5$, $E = 134722kN/m^2$, $G = 48529kN/m^2$ in order to compare the results with those obtained by the FEM using the NASTRAN code for thin orthotropic plates with negligible thickness ($h = 0.004m$). The obtained results for the central deflection $\bar{w}_0$ versus the load $\bar{g}$ are shown in Fig.2. Afterwards the membrane was prestressed as shown in Fig.3 ($\tilde{u} = \tilde{v} = 0.05m$) and analyzed under a uniform load. The employed data are $\bar{g} = 0.06$, $h = 0.002m$; $\lambda = 1.0,\ 1.5,\ 2.0$ and $\sqrt{E_1 E_2} = E = 110000kN/m^2$, $G = E/2(1 + \nu_1\sqrt{\lambda})$. Results for the response of the membrane are shown in Fig.4 through Fig.7.

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4.2 Square heterogeneous membrane

A square heterogeneous isotropic ($\lambda = 1$) membrane, subjected to a uniform load, has been analyzed ($N = 100$, $M = 49$). The membrane was prestressed as in example 4.1. The heterogeneity results from the variable thickness of the membrane. The employed data are $a = 5.0m$, $g = 3kN/m^2$, $E_1 = E_2 = E = 110000kN/m^2$, $\nu_1 = \nu_2 = 0.3$. Three cases of thickness variation have been studied (i) $h = h_0[1 + 7r^2/a^2]$, $r = (x^2 + y^2)^{1/2}$, $h_0 = 0.0005m$ (ii) $h = 13h_0/6$ and (iii) $h = h_0[12 - 7r^2/a^2]/5$. In all three cases the volume V of the material has been kept unchanged, that is $V = 13h_0a^2/6$. The obtained results for the deflections and stresses along $y = 0$ are shown in Fig.8 and Fig.9. It should be noted that the case (i) gives large deflections, while the two others have no significant difference.

4.3 Membrane of arbitrary shape

In this example, the heterogeneous orthotropic membrane of arbitrary shape shown in Fig.10 was analyzed ($N = 80$, $M = 61$). Its boundary is defined by the curve $r = 5|\sin\theta|^3 + 6|\cos\theta|^3$, $0 \le \theta \le 2\pi$. The membrane is prestressed by $u_n = 0.05m$ in the direction normal to the boundary while $u_t = 0$ in the tangential direction. The employed data are $h = 0.002m$, $g = 3kN/m^2$, $E_1 = E/\sqrt{\lambda}$, $E_2 = E\sqrt{\lambda}$, $\nu_1 = 0.3$, $\nu_2 = \lambda\nu_1$ and $G = E/2(1 + \nu_1\sqrt{\lambda})$ where $E = 110000 + kr^2$, $r = (x^2 + y^2)^{1/2}$ and k constant. The computed deflections and the contours of the principle stress resultants N_1 for various values of k and λ are shown in Fig.11 through Fig.13.

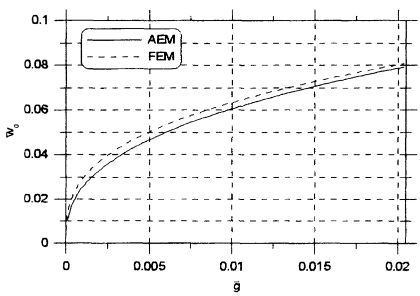

Figure 2: Central deflection versus load

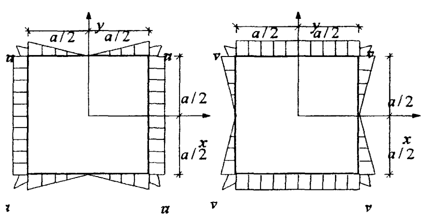

Figure 3: Prestress by imposed displacements

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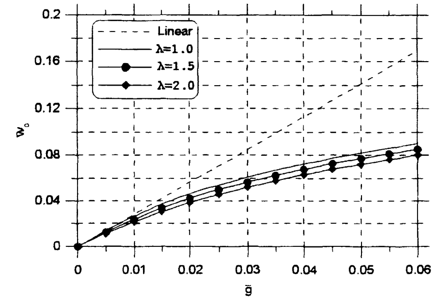

Figure 4: Central deflection versus load

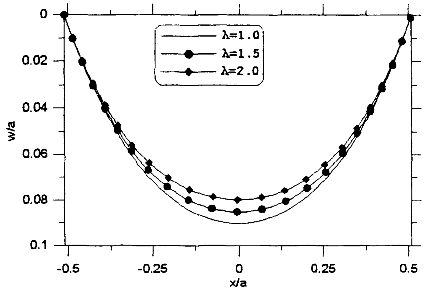

Figure 5: Deflection profiles at $y = 0$

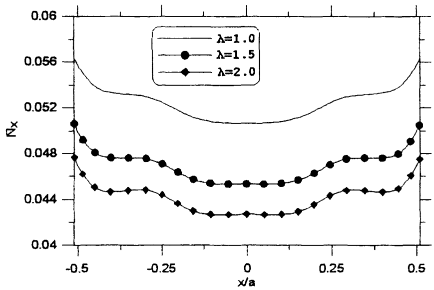

Figure 6: Variation of $\overline{N}_x$ at $y = 0$

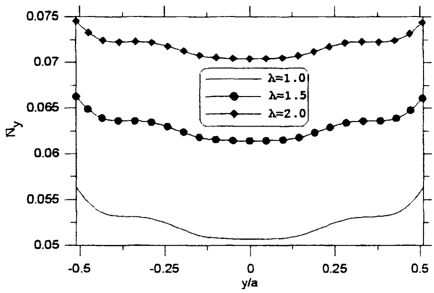

Figure 7: Variation of $\overline{N}_y$ at $x = 0$

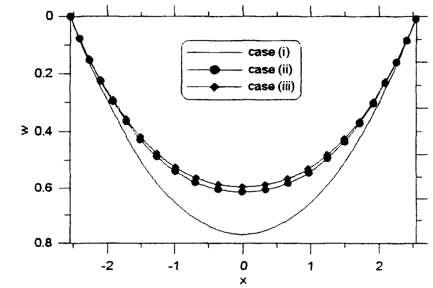

Figure 8: Deflection profiles at $y = 0$

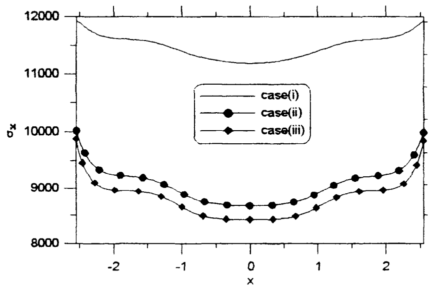

Figure 9: Variation of σ_x at $y = 0$

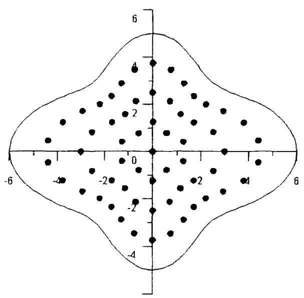

Figure 10: Membrane of arbitrary shape and nodal points

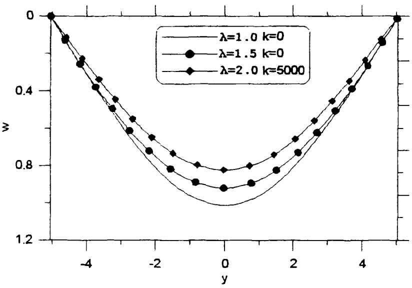

Figure 11: Deflection profiles at $x = 0$

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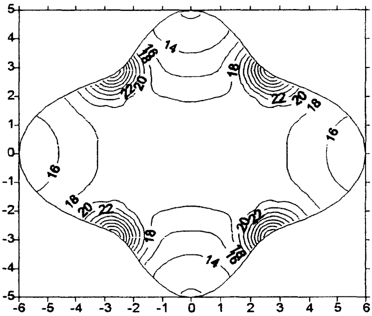

Figure 12: Contour of N_1 ($\lambda = 2$, $k = 0$)

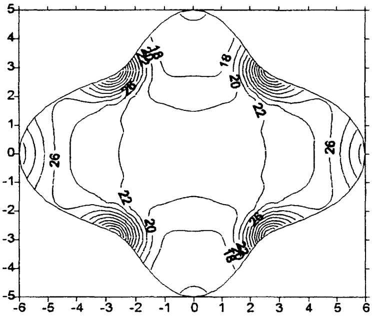

Figure 13: Contour of N_1 ($\lambda = 2$, $k = 5000$)

5 Conclusions

In this paper a boundary-only method has been presented for large deflection analysis of initial flat heterogeneous anisotropic elastic membranes. The method is based on the concept of the analog equation. From the presented analysis and the numerical examples the following main conclusions can be drawn.

(i) As the method is boundary-only it has all the advantages of the BEM, i.e. the discretization and integration are performed only on the boundary.

(ii) The deflections and the stress resultants are computed at any point using the respective integral representation as mathematical formulas.

(iii) Accurate numerical results for the displacements and the stress resultants are obtained.

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