Advances in Fluid Mechanics III, C.A. Brebbia & M. Rahman (Editors)
© 2000 WIT Press, www.witpress.com, ISBN 1-85312-813-9

Nonlinear hydrodynamic loading on offshore structures

M. Rahman
Department of Engineering Mathematics, DalTech, Dalhousie University, P.O. Box 1000, Halifax NS, B3J 2X4, Canada

Abstract

This paper is mainly concerned with the investigation of the second order wave loading on offshore structures. The diffraction theory of hydrodynamics is used to predict the wave loading. Lighthill's[2] second order theory has been extended to include the investigation to the shallow water case. Wave forces on cylindrical structures are calculated using Lighthill's theory, and the numerical results are compared with the available experimental data. The present theory encompasses the entire spectrum of ocean waves, from deep to shallow water cases, and makes a significant step forward for determining the wave loads on submerged structures.

1 Introduction

The problems of hydrodynamic waves and wave loading estimation for offshore structures have received considerable mathematical and engineering attention due to their many practical applications to ocean engineering. The United States Commission on Marine Science and Engineering Resources has predicted that underwater structures will be built at an increasing rate in the near future. In Canada, such structures will be built in connection with the recovery and production of oil and gas. Therefore, it is necessary to be able to accurately predict forces on submerged structures; this will enable the dynamic characteristics of these structures to be determined.

The wave loading estimations for offshore structures are based on the well-known "Morison Equation", which involves both drag and inertia forces. Morison et al[3] presented an equation for calculating the total force on an object under the influence of gravity waves. The equation expresses the

Advances in Fluid Mechanics III, C.A. Brebbia & M. Rahman (Editors)
© 2000 WIT Press, www.witpress.com, ISBN 1-85312-813-9

total hydrodynamic drag D as a sum of an inertia force, $\rho C_M VU$ with $C_M = 1 + \frac{M_a}{\rho V}$ where M_a is the virtual mass of the body, associated with the irrotational flow component, and the viscous drag force $\frac{1}{2}\rho C_D AU^2$, related to the vortex-flow component. The two characteristics of the Morison equation are the inertia force, which is linear in velocity U, and the drag force, which is nonlinear in velocity U.

The present paper is mainly concerned with the investigation of the nonlinear wave loading on offshore structures using the diffraction theory of hydrodynamics. It extends Lighthill's second-order theory, which is applicable only to deep ocean waves, to the cases of finite depth waves. All linear and nonlinear wave loadings on vertical piles immersed partially in an ocean of arbitrary uniform depth are calculated explicitly. It is demonstrated in this paper that all nonlinear boundary conditions including the radiation condition are satisfied by the scattered velocity potential, a considerable improvement over previous theories. Comparison is made between the linear and nonlinear forces. The results are presented such that the deep ocean case considered by Lighthill becomes a limiting case of the analysis. The present theory covers the entire spectrum of ocean waves, from the deep to shallow water cases, and makes a significant step forward in the evaluation of nonlinear wave loadings on submerged structures.

2 Mathematical analysis

We know that the irrotational fluid flow pressure distribution can be obtained from Bernoulli's equation, which is

$$\frac{P}{\rho} + gz + \frac{\partial\phi}{\partial t} + \frac{1}{2}[(\frac{\partial\phi}{\partial x})^2 + (\frac{\partial\phi}{\partial y})^2 + (\frac{\partial\phi}{\partial z})^2] = 0. \tag{1}$$

We can define the following three pressure distribution from equation (1):

I. The hydrostatic pressure distribution, which is given by

$$P_0 - \rho g z \ , \tag{2}$$

at height z above the level $z = 0$, where the hydrostatic pressure is P_0.

II. The dynamic pressure distribution of Bernoulli, given by

$$\frac{1}{2}\rho(U^2 - q^2) \ , \tag{3}$$

at a point where the fluid speed is q, and U is defined in the next paragraph.

III. The transient pressure distribution, given by

$$-\rho\frac{\partial\phi}{\partial t} \ . \tag{4}$$

Advances in Fluid Mechanics III, C.A. Brebbia & M. Rahman (Editors)

There is a certain resultant force with which each of these pressure distributions acts on a body in a stream of homogeneous fluid of variable speed U. U is simply the fluctuating fluid velocity that would be found where the body is if the body were absent. At the surface $z = \eta$, equation (1) takes the following form

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z} + \frac{\partial}{\partial t}(\nabla\phi)^2 + \frac{1}{2}\vec{q}.\nabla(\nabla\phi)^2 = 0 \ , \quad \text{at} \quad z = \eta. \tag{5}$$

Retaining up to the second-order term in the variable ϕ in equation (5), we obtain

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z} = -\frac{\partial}{\partial t}(\nabla\phi)^2 \ . \tag{6}$$

Equation (6) was obtained by Lighthill.

3 Second order wave loading

Lighthill in 1979 demonstrated that a second-order contribution to the irrotational flow loading on a structural component consists of the resultant force F_d of the dynamic pressure (II) associated with the linear velocity potential ϕ_l. This can be written as an integral, namely

$$F_d = \int_S \frac{1}{2}\rho(\nabla\phi_l)^2 n_x dS, \tag{7}$$

over the surface of the body, where n_x is the direction cosine between the outward normal and the direction of the force component F_d being determined.

Another second-order force is associated with the linear velocity potential if the structural component penetrates the surface. The whole additional second-order horizontal x-component of force acting at the waterline w can be written as

$$F_w = \oint_w \frac{\rho}{2g}(\frac{\partial \phi_l}{\partial t})^2 ds, \tag{8}$$

where the integrand is the horizontally resolved force per unit length acting at the waterline, and the integral gives its resultant in the x-direction.

Following Lighthill's technique, the quadratic potential, ϕ_q, is uniquely determined as the potential of the linearized wave motion generated in the presence of the stationary structure due to a fluctuating pressure distribution

$$\rho[(\nabla\phi_l)^2 + \frac{1}{2}(\tanh^2 kh - 1)(k\phi_l)^2] \tag{9}$$

applied at the surface.

Expression (9) reduces to Lighthill's expression (7) for deep water case (note that $\lim_{kh\to\infty} \tanh kh \simeq 1.$).
Thus, the quadratic force F_q takes the integral form

$$F_q = -\int_S W\rho[(\nabla\phi_l)^2 + \frac{1}{2}(\tanh^2 kh - 1)(k\phi_l)^2]dS \tag{10}$$

Here, W represents the vertical velocity distribution and dS is an elementary area of the free surface S. This result is obtained by using Green's theorem together with a proper application of the radiation condition to the velocity fields.

The linear-diffraction theory yields the potential function ϕ_l, from which a force F_l may be computed by linear analysis. By adding to F_l, three more terms F_d, F_w, and F_q, given by (7), (8), and (10), more accurate values of the horizontal x-component of the force can be obtained. The total fore exerted on the body $\vec{F}$ and the moment of this force $\vec{M}$ in vector notation are then given by

$$\vec{F}(t) = -\int_S P\vec{n}dS \tag{11}$$

$$\vec{M}(t) = -\int_S P(\vec{r}\times\vec{n})dS \tag{12}$$

where $\vec{n}$ is the outward normal vector on S, P is the pressure and $\vec{r}$ is the vector from the point about which moments are taken.

4 Calculation of F_q in finite water depth

Putting

$$\phi_q = Re[\Phi_q e^{2i\omega t}] \tag{13}$$

and

$$\phi_l = Re[\Phi_l e^{i\omega t}] \tag{14}$$

we obtain the equation

$$(\frac{\partial\Phi_q}{\partial z} - K\Phi_q)_{z=0} = -\frac{2i\omega}{g}(\nabla\Phi_l)^2 + \frac{i\omega}{g^2}\Phi_l(-\omega^2\frac{\partial\Phi_l}{\partial z} + g\frac{\partial^2\Phi_l}{\partial z^2})_{z=0} \tag{15}$$

Advances in Fluid Mechanics III, C.A. Brebbia & M. Rahman (Editors)

with $K = \frac{4\omega^2}{g}$. Now, let the potential due to a unit translation oscillation of the body be

$$Re|\psi e^{2i\omega t}| \ .$$

Then, on the body surface S,

$$(\frac{\partial\psi}{\partial n})_S = n_x, \qquad (\frac{\partial\Phi_q}{\partial n})_S = 0, \tag{16}$$

where n_x is the x-component of a unit inward normal to S. Also,

$$(\frac{\partial\psi}{\partial z} - K\psi)_{z=0} = 0 \ , \tag{17}$$

and, like Φ_q, ψ satisfies the radiation condition.

This allows us to apply Green's theorem, which gives

$$\int\int_v \{\Phi_q\nabla^2\psi - \psi\nabla^2\Phi_q\}dv = \int_{SUz=0}\{\Phi_q\frac{\partial\psi}{\partial n} - \psi\frac{\partial\Phi_q}{\partial n}\}dS \tag{18}$$

taken over the boundaries of the fluid (S and $z = 0$). Here, n is a normal outward from the fluid.

Since Φ_q and ψ both satisfy Laplace's equation, we have

$$\int_{SUz=0}[\Phi_q\frac{\partial\psi}{\partial n} - \psi\frac{\partial\Phi_q}{\partial n}]dS = 0 \ . \tag{19}$$

After performing the indicated integration in equation (19) and applying the conditions (16) and (17), we can write

$$\int_{z=0}\psi(\frac{\partial\Phi_q}{\partial z} - K\Phi_q)dx\ dy = \int_S \Phi_q n_x dS. \tag{20}$$

Now, the force F_q can be written as

$$\begin{aligned} F_q &= \int_S(-\rho\frac{\partial\phi_q}{\partial t})n_x ds \\ &= Re[-\rho 2i\omega e^{2i\omega t}) \int_S \Phi_q n_x dS], \end{aligned} \tag{21}$$

and is given by

$$F_q = Re[-\int_{z=0} \rho W\{(\nabla\phi_l)^2 + \frac{1}{2}(\tanh^2 kh - 1)(k\phi_l)^2\}dxdy] \quad (22)$$

Here

$$W = (\frac{\partial\psi}{\partial z})_{z=0} \quad (23)$$

is the vertical velocity on the free surface associated with the unit translational oscillation of the body. For deep water waves, this result reduces to Lighthill's equation (14).

5 Exact calculations for second order wave loads

For a diffracted wave, the total horizontal force, F_l, can be obtained from the following formula:

$$F_l = \int_{\theta=o}^{2\pi}\{\int_{z=-h}^{0}(-\rho\frac{\partial\phi_l}{\partial t})dz\}_{r=b}(-\cos\theta)bd\theta \ . \quad (24)$$

After inserting the expression for ϕ_l and performing the indicated integration, F_l can be written as:

$$F_l = C_M\frac{\rho g H\pi b^2}{2}\tanh kh\cos(\omega t - \beta) \quad (25)$$

where

$$C_M = \frac{4}{\pi k^2 b^2\sqrt{J_1'^2(kb) + Y_1'^2(kb)}} \quad (26)$$

$$\beta = \tan^{-1}\frac{J_1'(kb)}{Y_1'(kb)} \quad (27)$$

To calculate the second order contributions F_d ,F_w and F_q, the calculations are summerized as follows:

The dynamic force, F_d, can be evaluated from the following formula:

$$F_d = \int_0^{2\pi}\{\int_{z=-h}^{0} -\frac{1}{2}\rho(\nabla\phi_l)^2dz\}(-\cos\theta)bd\theta \quad (28)$$

Advances in Fluid Mechanics III, C.A. Brebbia & M. Rahman (Editors)
© 2000 WIT Press, www.witpress.com, ISBN 1-85312-813-9

Now the total dynamic force F_d, may be written as

$$F_d = \frac{\rho g H^2}{2\pi b k^2}\sum_{l=0}^{\infty}[(1-\frac{2kh}{\sinh 2kh})+\frac{l(l+1)}{k^2b^2}+(1+\frac{2kh}{\sinh 2kh})] \times [E_l-(-1)^l\{C_l\cos 2\omega t - S_l \sin 2\omega t\}] \quad (29)$$

where

$$\left.\begin{aligned} E_l &= (J_l'Y_{l+1}'-J_{l+1}'Y_l')/T_l \\ C_l &= (Y_l'J_{l+1}'+Y_{l+1}'J_l')/T_l \\ S_l &= (Y_l'Y_{l+1}'-J_l'J_{l+1}')/T_l \\ T_l &= (J_l'^2+Y_l'^2)(J_{l+1}'^2+Y_{l+1}'^2), \end{aligned}\right\} \quad (30)$$

the Bessel function arguments being kb. The quadratic force F_q is rewritten as follows

$$F_q = Re\{-\frac{\rho}{2}e^{2i\omega t}\int_b^{\infty} rdr \int_0^{2\pi}(\frac{\partial\psi}{\partial z})_{z=0} \times [(\nabla\Phi_l)^2+\frac{k^2}{2}(\tanh^2 kh-1)(\Phi_l)^2]_{z=0}d\theta \quad (31)$$

Invoking the expression for Φ_l and the expression for $(\frac{\partial\psi}{\partial z})_{z=0}$ and after some algebraic reduction, we obtain

$$\begin{aligned} F_q = & Re[\frac{\rho g\pi H^2}{2k}\coth\ kh\int_{kr=kb}^{\infty} e^{2i\omega t}\sum_{n=0}^{\infty}[2\vec{Q}'_n\vec{Q}'_{n+1}+(\frac{2(n+1)n}{k^2r^2} \\ & - 1+3\tanh^2 kh)\vec{Q}'_n\vec{Q}'_{n+1}]\{(-1)^n(C_n \\ & + iS_n)\}\{\frac{H_1^{(2)}(\kappa r)}{H_1^{(2)'}(\kappa b)}\frac{\sinh^2\kappa h}{\kappa h+\sinh\kappa h\cosh\kappa h} \\ & - \sum_{j=1}^{\infty}\frac{K_1(m_jr)}{K_1'(m_jb)}\frac{\sin^2 m_jh}{m_jh+\sin m_jh\cos m_jh}\}\kappa r\ d(\kappa r)]. \end{aligned} \quad (32)$$

where K_1 is the modified Bessel function of second kind with order one, and

Advances in Fluid Mechanics III, C.A. Brebbia & M. Rahman (Editors)

$$
\begin{aligned}
A_n(kr) &= Q_n(kr)e^{-i\alpha_n} \\
Q_n(kr) &= \frac{\vec{Q}_n(kr)}{\sqrt{J_n'^2(kb) + Y_n'^2(kb)}} \\
\vec{Q}_n(kr) &= J_n(kr)Y_n'(kb) - J_n'(kb)Y_n(kr) \\
\alpha_n(kb) &= \tan^{-1}\frac{J_n'(kb)}{Y_n'(kb)}
\end{aligned}
$$

6 Results and conclusion

An exact second-order diffraction theory has been developed in this paper which extends Lighthill's theory for deep water waves so as to include shallow water waves. The exact expression for the velocity potential in the case of a surface-piercing circular cylinder has been used. The theory has been tested by comparison with results of previous theory and experimental data. The maximum horizontal force is expressed as the average of the absolute values of the maximum positive force and the maximum negative force.

The present theory differs from that of Rahman and Heaps [4] in the way in which the quadratic force, F_q, has been obtained. We have used Green's function formulation, which makes use of linear velocity potential, contrasting with their approach, which requires tedious algebraic calculation of the quadratic velocity potential. It has been found that both theories lead to identical results. However, the integral representation of the quadratic force differ slightly in form, which is attributed to the different theoretical developments. The drift force has been defined as the sum of the steady state parts of the dynamic and waterline forces.

Figure 1 gives a plot of C_M and β as defined in section 6. In Figure 6, comparison is made between the results of this theory and the experimental data of Chakrabarti [1]. Again, the present theory correlates well with the experimental data for the diffraction parameter kb greater than unity. However, for $kb < 1$, the experimental data seem not to correlate well with the theory. The reason behind this discrepancy cannot be determined with the present theory and therefore further investigation is warranted.

We have used the Simpson's integration scheme in evaluating the double integral in the quadratic force evaluation. The IMSL library was used in evaluating the Bessel functions of different orders arising in the integrand. In the numerical calculations it is found that the r-integrand is highly oscillatory. The integration has been carried out from the surface of the cylinder over a large distance, B, as indicated in the approximate theory for small kb. Extensive numerical tests indicate that beyond the limiting B, the contribution of the integral is insignificant.

Acknowledgement

The author is thankful to Natural Sciences and Engineering Research Council of Canada for their financial support. Thanks are also extended to Isaac Mulolani for his help in preparation of the final manuscript. This paper has been dedicated to the memory of Sir James Lighthill, FRS who died on 17 July 1998.

References

[1] Chakrabarti, S.K. Second order wave force on large vertical cylinder. *Engg. Div., ASCE*, **101**, WW3, Proc. paper 11476, 311-317, 1975.

[2] Lighthill, M.J. Waves and hydrodynamic loading, *Proc. 2nd int. conf. on behavior of offshore structures*, **1**, 1-40, 1979.

[3] Morison, J.R. O'Brien, M.P., Johnson, J.W. and Schaaf, S.A. The forces exerted by surface wave on piles. *J. Petrol. Technol.*, **189**, 149-154, 1950.

[4] Rahman, M. and Heaps, H.S. Wave forces on offshore structures: Nonlinear wave diffraction by cylinders. *J. Phys. Oceano.*, **13**, 2225-2235, 1983.

Advances in Fluid Mechanics III, C.A. Brebbia & M. Rahman (Editors)
© 2000 WIT Press, www.witpress.com, ISBN 1-85312-813-9

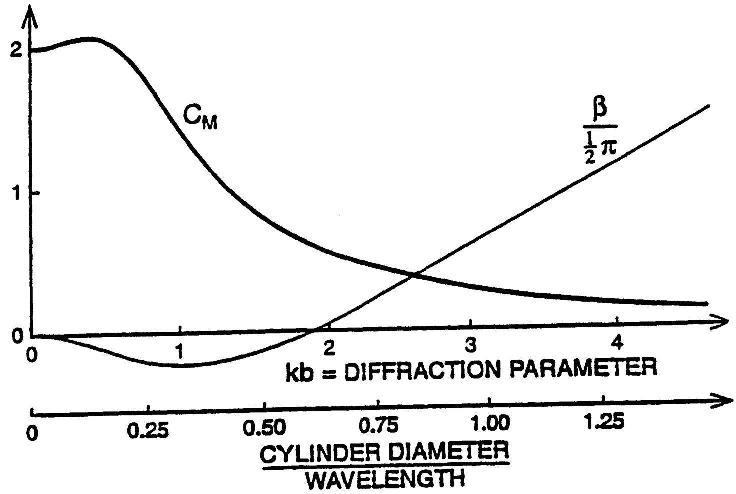

Figure 1: Graphs of C_M and *2* plotted versus *kb*

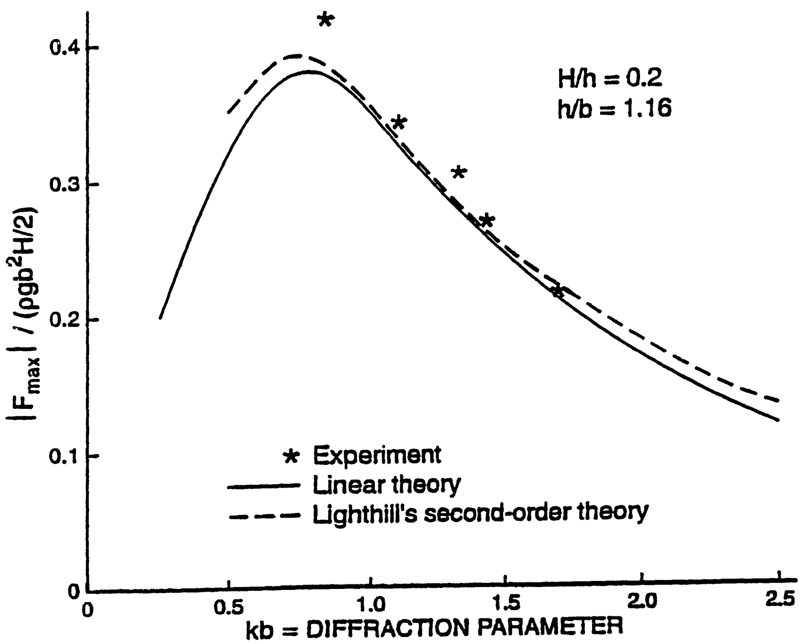

Figure 2: Comparison of linear and second order wave forces with experimental data.