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This book is dedicated to Sir James Lighthill, FRS who gave the author tremendous inspiration in applied mathematics during his days at Imperial College London.

“We make a living by what we get
but we make a life by what we give”

Sir Winston Churchill

Contents

Preface

The generalized function is one of the important branches of mathematics that has enormous application in practical fields. Especially, its applications to the theory of distribution and signal processing are very much noteworthy. The method of generating solutions is the Fourier transform, which has great applications to the generalized functions. These two branches of mathematics are very important for solving practical problems. While I was at Imperial College London (1966–1969), I attended many lectures delivered on fluid mechanics topics by Sir James Lighthill, FRS. At that time I was unable to understand many of the mathematical ideas in connection with the generalized function and why we need this abstract mathematics in the applied field. I tried to follow Lighthill's book *An Introduction to Fourier Analysis and Generalized Functions*, published by Cambridge University Press, 1964. His book is very compact (only 79 pages) and extremely stimulating, but he has written it so elegantly that unless one has good mathematical background, the book is very hard to follow. I understand that a non-expert reader will find the book very hard to follow because of its compactness and too many cross-references. Mathematical details are very minimal and he sequentially explains from one step to another skipping many intermediate steps by the cross references. Lighthill followed the ideas originally described by Professor George Temple's *Generalized Functions, Proc. Roy. Soc. A,* **228***, 175–190, (1955)*. Lighthill kept the theory part as described by Temple. In Dalhousie University I used to give a course on *Mathematical Methods and their Applications* to the undergraduate and graduate students for several years. I used Fourier transforms and generalized functions in that course. To make it understandable to the student I had to take recourse to some engineering textbooks where the applications are found in this subject. I followed some engineering application of generalized functions and its solution technique using the Fourier transform method.

This book grew partly out of my course given to the undergraduate and graduate students at Dalhousie University, Halifax, Nova Scotia, Canada; and partly from reading the books by Temple and Lighthill. This book explains clearly the intermediate steps not found in any other book. The book leans heavily towards Lighthill's book, but I have bridged the gap of mathematical deductions by clearly manifesting every important step with illustrations and mathematical tables. I think a layman can also follow my book without much difficulty. I must admit that this book is written in such a way as if I have revisited Lighthill's book *An Introduction to Fourier Analysis and Generalized Functions*. This book, hopefully, will be useful to the non-expert and also the experts alike. With this intention, the book is prepared in my own way collecting some additional material from some other textbooks including Professor D.S. Jones' book on *Generalized Functions*, published by McGraw-Hill Book Company, New York, 1966. I have borrowed some ideas from Professor Jones' book. Specially, I borrowed some important practical unsolved examples that I solved myself for the benefit of the reader. It is my hope that the reader will gain some insight about this important but esoteric mathematical subject.

The first chapter of the book deals with the introductory concept of Fourier series, Fourier integrals, Fourier transforms and the generalized function. The theoretical development of the Fourier transform is described and the first generalized function is defined with some illustrations. Some important examples are manifested in this chapter. Some interesting exercises are included at the end of the chapter.

Chapter 2 deals with the formal definition of the generalized function. A clear-cut definition of a good function and a fairly good function as illustrated by Lighthill is demonstrated in this chapter. The difference between an ordinary function and a generalized function is given with some examples. Even and odd generalized functions are clearly defined. The chapter ends with some useful exercises.

Chapter 3 consists of Fourier transforms of particular generalized functions. This chapter deals with the integral power of an algebraic function, non-integral powers, the Fourier transforms of $x^n \ell n|x|$, $x^{-m}\ell n\ |x|$, $x^{-m}\ell n|x|sgn(x)$ together with the summary of results of Fourier transforms. The chapter concludes with some exercises.

Asymptotic estimation of Fourier transforms are discussed in details in Chapter 4. First we have defined the Riemann-Lebesgue lemma which is important to obtain the asymptotic value of a generalized function. The asymptotic expression of the Fourier transform of a function with a finite number of singularities is discussed. We demonstrated solutions of some generalized functions using asymptotic expressions. Fourier transforms play a major role. Some important numerical solutions of some integrals are listed in Table 4.1. Whereas Table 4.2 contains a short list of Fourier transforms of 18 important generalized functions at a glance. The chapter ends with some important exercises.

Chapter 5 contains the Fourier series as a series of generalized functions. We demonstrated how to evaluate the coefficients of a trigonometric series. Some practical examples such as Poisson's summation formula and the asymptotic behaviour of the coefficients in a Fourier series are illustrated. This chapter concludes with some exercises.

We conclude the book (Chapter 6) with an important topic concerning the fast Fourier transform. It is a numerical procedure which is fast, accurate and efficient to determine the Fourier coefficients that are the Fourier transforms using an algorithm developed by Cooley and Tukey in 1965. Some preliminary studies of the Fourier transform with ample examples are also demonstrated in this chapter by using analytical and graphical methods. We have not reiterated the algorithm of Cooley and Tukey, rather we have given a numerical view of how it works, citing a practical example in the study of wave energy spectrum density as illustrated elegantly by Chakrabarti (1987). A handful of exercises are included and some references are cited at the end of the chapter.

The book concludes with three appendices. Appendix A deals with Fourier transforms of some important generalized functions. Appendix B is concerned with some important properties of Dirac delta $\delta(x)$ functions and Appendix C contains a comprehensive list of some important references concerning with the generalized functions and the application of the fast Fourier transform for further reading. A subject index is also included at the end of the book.

While it has been a joy to write such a comprehensive book for a long period of time, the fruits of this labour will hopefully be in learning of the enjoyment and benefits realized by the reader. Thus the author welcomes any suggestions for the improvement of the text.

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This book is primarily derived from Lighthill's book on *Introduction to Fourier analysis and generalised functions* published by Cambridge University Press in 1958 and subsequently reprinted in 1964. Thus, Cambridge University Press deserves my appreciations for the use of ideas and concepts which help me develop the present manuscript. WIT Press is gratefully acknowledged for its superb job in producing such a beautiful book.