Efficient regular perturbation solutions for beams subjected to thermal imperfections: a case study

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Abstract

A mesh reduction (regular) perturbation technique was developed to overcome inefficient and unviable analytical and "brute force" numerical solutions for structures with imperfections and for imperfection sensitive structures. Using this perturbation technique, a case study is presented to determine the effects of uncontrolled deviations in temperature on the stability of beam on elastic foundation. The study further explores the effects of imperfections on beams for five independent imperfection patterns, namely variability in initial shape, modulus of elasticity, moment of inertia, foundation stiffness, temperature, and their combined effects. The study demonstrates thermal imperfections behave in the same manner as other non-shape imperfections, while shape imperfections appear to be most sensitive. When thermal and shape imperfections were combined, all other imperfections were shown to have diminished effects.
Keywords: stability, thermal imperfections, beam on elastic foundation, regular perturbation, eigenvalue.

1 Introduction

Micro-sensors and devices are not only small but are also fragile. Small imperfections in shape, materials, and operating conditions could severely limit their use. How these small devices behave in less than perfect conditions is of great interest. How will the stability of these devices be affected?

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Two predominant classical stability analyses are differential equations and energy method approaches. In theory these approaches have worked well for conservative structures that are insensitive to imperfections, but in practice no structure has perfect geometry and the applied load may not be concentric. Tolerances account for irregularities in structural members.

Cylindrical shells and beams on elastic foundations are two structure types that are sensitive to imperfections [1, 10]. Structural imperfections are defined as any small and unintended deviations or variations from the perfect structure [7, 13]. If the structure is sensitive to imperfections, the neighboring equilibrium position exists at loads smaller than the critical load; this equilibrium position is unstable [9]. The nature of structural imperfections is generally small and unavoidable. Imperfections considered in this study include variability and combined effects in shape, modulus of elasticity, moment of inertia, foundation stiffness, and temperature.

2 Numerical method

In 1945, Koiter showed that imperfection effects on structures caused differences between theoretical and experimental results [5]. Koiter also examined the interaction of various buckling modes and analyzed imperfection sensitivity; however, this investigation was limited to shape imperfection only [13]. For medium and high imperfection-sensitive structures, the buckling load for the perfect structure extends below the bifurcation load by as much as 70%. Palassopoulos concluded that Koiter's analysis was fundamentally inadequate for imperfection sensitivity [8] and proposed the critical imperfection magnitude (CIM) method [7]. His theory was based on the expansion of potential energy without any limitations to shape imperfection.

CIM is based on the second-order expansion of the potential energy and the fourth-order expansion of a kinematically admissible set of generalized coordinates. For the present nonlinear problem, the second-order expansion of potential energy has given good results for the inextensional beam on elastic foundation (BEF) [7, 13].

A "perfect" structure represents those beams with zero imperfections, while "actual" structures contain imperfections of varying degrees. First, the potential energy of the "perfect" structure, V_0, is expanded in terms of the kinematically admissible set of generalized coordinates q_j, j = 1, 2, …, M.

$$V_0 = v_0 + a_{0j}q_j + b_{0jk}q_jq_k + c_{0jkl}q_jq_kq_l + \ldots \tag{1}$$

The subscript, 0, in variables and coefficients denotes a perfect structure. For the "actual" structure its potential energy, V, is expended about V_0, which is the potential energy for the corresponding "perfect" structure.

$$V = V_0 + \varepsilon V_1 + \varepsilon^2 V_2 + \ldots \tag{2}$$

$$V_1 = v_1 + a_{1j}q_j + b_{1jk}q_jq_k + c_{1jkl}q_jq_kq_l + \ldots \tag{3}$$

$$V_2 = v_2 + a_{2j}q_j + b_{2jk}q_jq_k + c_{2jkl}q_jq_kq_l + \ldots \tag{4}$$

The universal perturbation parameter is denoted by ε, which must be sufficiently small for the convergence of power series expansion of V. The same parameter, ε, is also used in the expansion of the material, load, and geometric parameters. For example, any geometric or material parameters, S(x), can always be expanded about its mean value, S_0, and an imperfection pattern, s(x), as $S(x) = S_0\,[1 + \varepsilon s(x)]$.

The coefficients $a_{(.)}$, $b_{(.)}$, $c_{(.)}$, and $d_{(.)}$ in eqns. (1)–(4) are chosen to be symmetric with respect to permutation of their indices. The application of symmetry helps to increase the computational efficiency for CIM. Then the potential energy is:

$$\begin{aligned} V = {} & (v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \ldots) + (a_{0j} + \varepsilon a_{1j} + \varepsilon^2 a_{2j} + \ldots)q_j \\ & + (b_{ojk} + \varepsilon b_{1jk} + \varepsilon^2 b_{2jk} + \ldots)q_j q_k \\ & + (c_{0jkl} + \varepsilon c_{1jkl} + \varepsilon^2 c_{2jkl} + \ldots)q_j q_k q_l \end{aligned} \tag{5}$$

The first and second variations, δV and $\delta^2 V$, of the potential energy yield equilibrium and stability equations. CIM requires two independent expansions: the potential energy expansion of the "actual" structures and expansion of the prebuckling equilibrium state q_j of the actual structure around the prebuckling equilibrium state q_{0j} of the perfect structure.

$$\begin{aligned} \delta V = {} & \{(a_{0j} + \varepsilon a_{1j} + \varepsilon^2 a_{2j} + \ldots) \\ & + 2\,(b_{0jk} + \varepsilon b_{1jk} + \varepsilon^2 b_{2jk} + \ldots)\, q_k \\ & + 3\,(c_{ojkl} + \varepsilon c_{1jkl} + \varepsilon^2 c_{2jkl} + \ldots)\, q_k q_l \end{aligned} \tag{6}$$

$$\begin{aligned} \delta^2 V = {} & \{2(b_{ojk} + \varepsilon b_{1jk} + \varepsilon^2 b_{2jk} + \ldots) \\ & + 6\,(c_{ojkl} + \varepsilon c_{1jkl} + \varepsilon^2 c_{2jkl} + \ldots)\, q_l \end{aligned} \tag{7}$$

The CIM approach will lead to formulation of a generalized eigenvalue problem in terms of ε. There are M eigenvalues in the solution, one for each buckling mode. The smallest eigenvalue is termed the critical imperfection magnitude, ε_{cr}, which is the smallest imperfection magnitude permissible prior to bifurcation.

The new eigenvalue equation is presented as:

$$\begin{bmatrix} \gamma_1 & \gamma_2 \\ I & 0 \end{bmatrix} \begin{Bmatrix} \delta q \\ \varepsilon \delta q \end{Bmatrix} = \frac{1}{\varepsilon} \begin{Bmatrix} \delta q \\ \varepsilon \delta q \end{Bmatrix} \tag{8}$$

$$\gamma_{1jk} = \frac{1}{\sqrt{b_{0jj} b_{0kk}}} \left[-b_{1jk} + \frac{3c_{0jkl} a_{1l}}{2b_{0ll}} \right] \tag{9}$$

$$\gamma_{2jk} = \frac{1}{\sqrt{b_{0jj} b_{0kk}}} \left[-b_{2jk} + \frac{3c_{1jkl} a_{1l}}{2b_{0ll}} + \frac{3c_{0jkl} a_{2l}}{2b_{0ll}} - \frac{3c_{0jkl} b_{1lm} a_{1m}}{2b_{0ll} b_{0mm}} \right.$$

$$+\frac{9c_{0jkl}c_{0lmm}a_{1m}a_{1n}}{8b_{0ll}b_{0mm}b_{0nn}}-\frac{3d_{0jklm}a_{1l}a_{1m}}{2b_{0ll}b_{0mm}}\Bigg] \tag{10}$$

where I and 0 represent identity and zero submatrices.

3 Beam on elastic foundation (BEF) formulation

BEF is simple yet sensitive enough to demonstrate the effects of smallest imperfections. The model has also been examined thoroughly in both deterministic and stochastic analyses [7, 13], and the model problem exhibits features readily found in many microdevices. Five independent imperfections will be considered herein, namely variability in initial shape, modulus of elasticity, foundation stiffness, moment of inertia, and temperature.

Consider a simply supported beam on elastic foundation. The beam is oriented in the standard right-hand system with the positive X-axis pointing to the right, the positive Y-axis pointing down. The beam has length (L), applied load (P), modulus of elasticity (E), moment of inertia (I), and elastic foundation stiffness (F). The temperature change is denoted by T, and the change exerts thermal loads on the beam. Applied loads cause the beam to deflect in the lateral direction; this deflection is represented by W. These variables are then transformed to dimensionless variables:

$$x=\frac{X}{L_P}=\frac{\pi X}{L};\, w=\frac{W}{L_P}=\frac{\pi W}{L};\, \varphi=\frac{L_P^4}{E_0 I_0}F=\frac{L^4}{\pi^4 E_0 I_0}F \tag{11}$$

$$\rho=\frac{L_P^2}{E_0 I_0}P=\frac{L^2}{\pi^2 E_0 I_0}P;\; v=\frac{L_P}{E_0 I_0}V=\frac{L}{\pi E_0 I_0}V \tag{12}$$

Imperfections fluctuate about their mean value. Shape imperfections (Type I) deviate from the zero mean and Type II imperfections (all others) fluctuate about some value other than zero and are determined by physical and geometric properties.

$$E(x) = E_0[1 + \varepsilon e(x)] \tag{13}$$

$$I(x) = I_0[1 + \varepsilon g(x)] \tag{14}$$

$$\varphi(x) = \varphi_0[1+ \varepsilon f(x) + \varepsilon C_t T_0 t(x)] \tag{15}$$

$$w_0 = \varepsilon h(x) \tag{16}$$

$$\varphi_0=\frac{L_p^4}{E_0 I_0}F_0=\frac{L^4}{\pi^4 E_0 I_0}F_0 \tag{17}$$

$$E(x)A(x)=\frac{E(x)I(x)}{L^2}=\frac{E_0 I_0[1+\varepsilon e(x)][1+\varepsilon g(x)]}{L^2} \tag{18}$$

$$T(x) = T_0[1 + \varepsilon t(x)] \tag{19}$$

In eqns. (13)–(19), E(x), I(x), φ(x), A(x), and T(x) represent elastic modulus, moment of inertia, foundation stiffness, cross section area, and temperature, respectively. Within the imperfection expressions, lowercase letters e(x), g(x), f(x), and t(x) represent deterministic imperfection patterns in elastic modulus, moment of inertia, foundation stiffness, and temperature change; mean values are represented by variables with zero subscripts.

The expression has three components: strain energy (V_B), potential energy of the applied load (V_P), and of the elastic foundation (V_F): $V = V_B + V_F + V_P$, where the first and second derivatives are denoted by single and double primes. Dividing the expression and its components by ($\pi E_0 I_0/L$) yield the dimensionless form of the potential energy. For the normalized beam, all integrals are evaluated for $x \in [0, \pi]$.

Before proceeding to the development of the coefficients of $a_{(.)}$, $b_{(.)}$, $c_{(.)}$, and $d_{(.)}$, the potential energy and imperfection patterns should be appropriately discretized. Since the modulus of elasticity (e), foundation stiffness (f), moment of inertia (g), and change in temperature (t) imperfections have no forced boundary conditions, a cosine series can be used to simulate imperfection patterns:

$$\bullet(x) = \sum_{j=1}^{N} \bullet_j \cos(jx) \tag{20}$$

where • is (e), (f), (g), or (t). Shape imperfections, on the other hand, are forced to be zero at the boundaries; a sine series is used:

$$h(x) = \sum_{j=1}^{N} h_j \sin(jx) \tag{21}$$

In eqns. (20-21), e_j, g_j, fj, t_j, and h_j are the deterministic imperfection amplitudes and N is the number of imperfection modes which are taken into account in the numerical computations. Similar to shape imperfections, lateral displacement modes are expanded in a sine series to meet the boundary conditions:

$$w(x) = \sum_{j=1}^{M} q_j \sin(jx) \tag{22}$$

where M is the number of displacement modes.

4 Numerical results

A numerical solution of the nonlinear eigenvalue problem is time-consuming stemming from a full coefficient matrix, which can be as large as 2M x 2M, where M is the number of buckling modes. Modifications have been made to the eigenvalue solver to allow a more efficient computation [13].

The coefficient matrix can be divided into four quadrants. The first quadrant contains the Type I coefficients (γ_{2jk}), and the second quadrant contains Type II (γ_{1jk}) coefficients. The third and fourth sectors contain identity and zero matrices, respectively. Solving combined imperfection types yields a full coefficient matrix. All imperfections were considered individually and collectively. Imperfection and buckling modes all converged in this study similar to a previous study [13]. Based on earlier studies [4, 7, 13], relatively stable/stiff soil parameter value, φ_0 of 225 is chosen for this study. Large values represent stiff foundation.

The classical buckling load, ρ_{cl}, refers to the applied load which causes the structure to buckle in the absence of imperfections. For actual structures the applied load, ρ, should be lower than the classical buckling load. It has been shown that the classical load for BEF is obtained by minimizing the expression of $(j^4 + \varphi_0)/j^2$ where j = 1, 2, 3, ..., M [7, 12, 13]. In this study, 99, 97, 95, and 90% of the classical load were used.

As for all imperfections, temperature imperfections vary about their mean value, T_0, described as the normalized average change in temperature. This normalization can be determined by dividing the mean change in temperature by the room temperature of 30°C. After analyzing the program output and considering its physical meaning, T_0 is set at -0.167, where a negative sign for T_0 means a reduction in temperature. Temperature reduction is important because it introduces the internal compression load to the BEF, which increases the applied load. Furthermore, a reduction in T_0 causes an increase in the foundation stiffness. The coefficient of thermal expansion, α, is set at 1.

The foundation stiffness factor, C_t, prescribes the percent increase in foundation stiffness when the temperature is reduced. C_t is normalized by the foundation stiffness, φ_0, and is chosen to be -0.0024. This value represents about a 0.04% increase in foundation stiffness per degree decrease in temperature based on Guyer and Brownell [3].

A cosine function is selected for Type II imperfections. The magnitude of the imperfection is chosen as 0.002, but imperfection magnitude does not affect ε_{RMS}.

$$\varepsilon_{RMS}^{(i)[\bullet]} = \varepsilon_{cr}^{(i)} \sqrt{\frac{1}{\pi} \int_0^{\pi} [\bullet(x)]^2 \, dx} \tag{23}$$

where $[\bullet]$ is e, k, h or t and i = 1, 2, …, M. The product of the imperfection magnitude ε_{cr} and RMS imperfection patterns should not be greater than 0.35 since it violates the nature of the perturbation approximation and are physically meaningless [13].

From fig. 1, five imperfection patterns were analyzed individually and compared to assess the relative importance to each other. Shape appears to be the most sensitive followed by temperature, foundation stiffness, moment of inertia, and modulus of elasticity. All Type II imperfections had ε_{RMS} similar in shape, orientation, and location. This result was anticipated since only γ_{1jk} in the potential energy was influenced by Type II imperfections.

A second observation from fig. 1 is that BEF was determined to be the most sensitive to shape imperfections and the least sensitive to change in temperature imperfections. The third observation is that the imperfection sensitivity for change in temperature imperfection increases when the magnitude increases of the mean value for change in temperature increases.

In fig. 2, ε_{RMS} values for temperature are shown. The mean value for change in temperature is chosen as -0.167 to demonstrate its effects. This figure shows that t(x) is the least sensitive by itself. The buckling strength of BEF is reduced when in the presence of other Type II imperfections. When all Type II imperfections are present together, ε_{RMS} can be reduced as much as 70%.

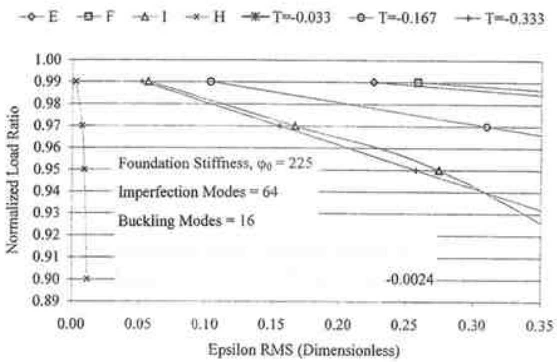

Figure 1: Direct comparison of e, f, g, h, and t.

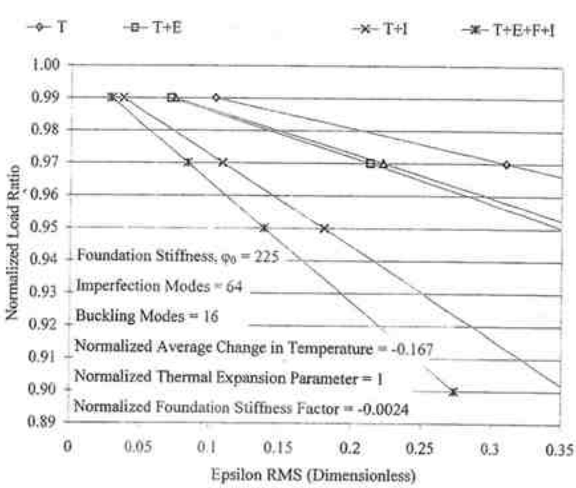

Figure 2: Effects on change in temperature ε^{t}_{RMS}

Fig. 3 represents ρ/ρ_{cr} plotted as a function of ε^{h}_{RMS} and ε^{t}_{RMS} for temperature and shape imperfections. The purpose of these two figures is to determine additive effects of Type II imperfections on the existing temperature and shape imperfections. These figures show no significant effects for the presence of other imperfections. Shape is the most dominant imperfection pattern.

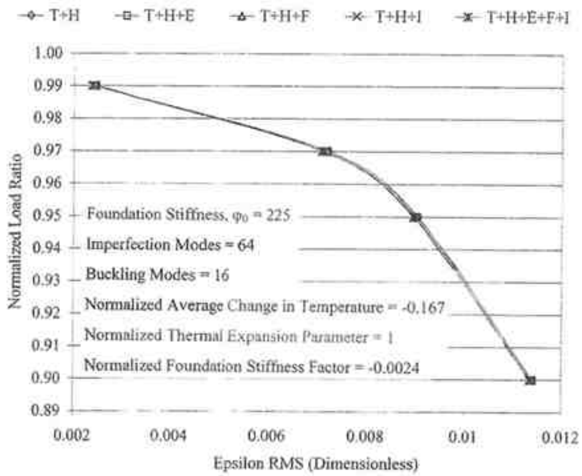

Figure 3: Effects on initial shape and temperature ε^{h}_{RMS} and ε^{t}_{RMS}.

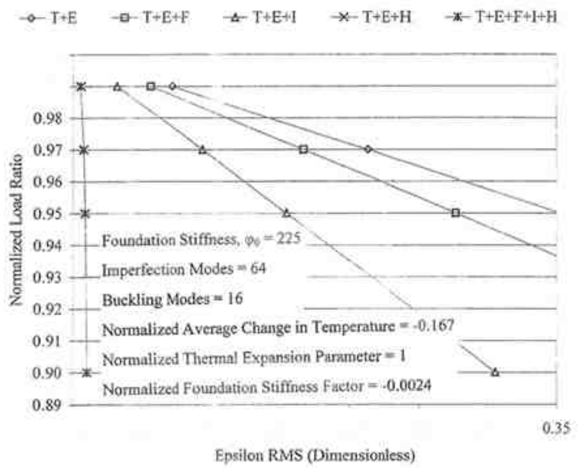

Figure 4: Effects on modulus of elasticity ε^{e}_{RMS}.

Figs. 4–6 represent ρ/ρ_{cr} plotted as a function of ε^{e}_{RMS}, ε^{f}_{RMS}, and ε^{g}_{RMS}. The presence of any Type II imperfections can reduce the ε_{RMS}, and the reduction can vary from 10 to 57%. The presence of Type I imperfections introduced a remarkable seventyfold increase in sensitivity.

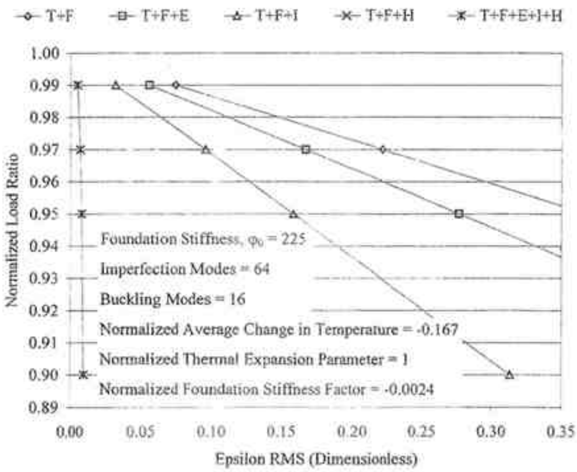

Figure 5: Effects on foundation stiffness ε^f_{RMS}.

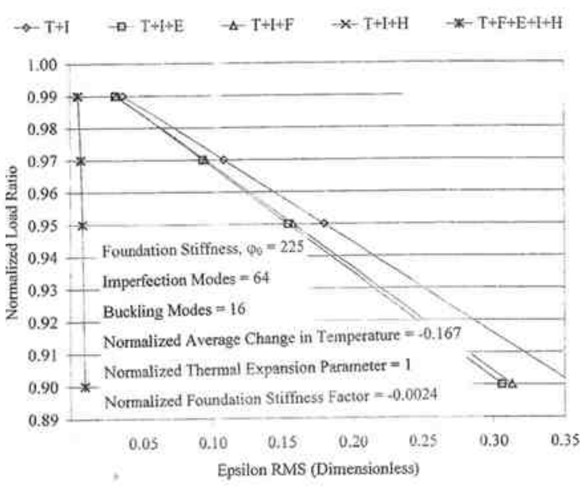

Figure 6: Effects on moment of inertia ε^g_{RMS}.

5 Conclusions

This study has considered imperfection effects on a beam on elastic foundation (BEF). The critical imperfection magnitude method has been used to analyze the significance of imperfections and their imperfection interactions on BEF.

Small, uncontrolled deviations in temperature from the mean value, T_0, have a negligible effect on the stability of BEF. In fact, temperature has the least impact when normalized average change in temperature is less than -0.033. If the mean value or amplitude of imperfections is large, temperature imperfections can have adverse effects. Normalized average change in temperature of -0.4, -0.233, -0.167, and -0.067 will reduce the effective buckling load to 90, 95, 97, and 99% of the classical value.

Even though the effects of temperature imperfection alone are insignificant, thermal imperfections can stimulate other material and geometric imperfections. When foundation stiffness imperfections are coupled with -0.167 variability in temperature, the load factor could be reduced to 71% of the classical value. Thermal imperfections behave in the same manner as other Type II imperfections, while shape imperfections appear to be most sensitive to BEF. When thermal and shape imperfections were combined, all other imperfections were shown to have diminished effects.

The critical imperfection magnitude method used in this study was strongly influenced by shape imperfection patterns. If all imperfection patterns for real structures are known, this deterministic study can be used to accurately determine the effects of those imperfections. If imperfection patterns are not known, stochastic methods can be used to accurately simulate imperfection patterns [4, 13].

References

[1] Almroth, B.O., Holmes, A.M.C., & Brush, D.O., An Experimental Study of the Buckling of Cylinders Under Axial Compression. *Experimental Mechanics*, pp. 263-270, 1964.

[2] Bazant, Z. & Cedolin, L. *Stability of Structures*. Oxford, 1991.

[3] Guyer, E.C. & Brownell, D.L., *Handbook of Applied Thermal Design*. McGraw Hill: New York, 1989.

[4] Hoffman, J.A., *Stability of Structures Subjected to Eccentric Load Imperfections*. M.S. thesis. Oklahoma State University, 1996.

[5] Koiter, W.T., *On the Stability of Elastic Equilibrium*. Ph.D. thesis. Delft, Holland, 1967.

[6] Langhaar, H.L., *Energy Method in Applied Mechanics*. Krieger: Melbourne, 1989.

[7] Palassopoulos, G.V., A New approach to the Buckling of Imperfection-Sensitive Structures. *Journal of Engineering Mechanics*, 119, pp. 850-869, 1993.

[8] Palassopoulos, G.V., Response Variability of Structures Subjected to Bifurcation Buckling. *Journal of Engineering Mechanics*, 118, pp. 1164-1183, 1992.

[9] Simitses, G.J., An Introduction to the Elastic Stability of Structures. Krieger: Melbourne, 1976.

[10] Tennyson, R.C., Muggeridge, D.B. & Caswell, R.D., Buckling of Cylindrical Shells Having Axisymmetric Imperfection Distributions. *AIAA Journal*, 9, pp. 924-930, 1971.
[11] Thompson, J.M.T. & Hunt, G.W., *Elastic Instability Phenomena*. Wiley: New York, 1984.
[12] Timoshenko, S.P. & Gere, J.M., *Theory of Elastic Stability*. McGraw-Hill: New York, 1961.
[13] Yeigh, B.W. *Imperfections and Instabilities*. Ph.D. thesis, Princeton University, 1995.