Multimodal torsional vibrations for the characterization of complex fluids

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Abstract

A fast and dynamic method to determine the viscosity of a Newtonian fluid can be achieved by measuring the damping value of a torsional resonator immersed in the fluid. For the investigation of complex fluids, the viscous as well as the elastic behaviour of the sample have to be observed. Since these properties vary with shear rate, several frequencies have to be analysed.

An approach is presented that allows to measure the viscoelastic parameters of a complex fluid via the resonance frequencies and damping values of a torsional rod vibrating at several vibration modes, i.e. at several frequencies.

The measurement is performed with a phase-locked loop (PLL) based control loop, which uses the characteristic value of the phase difference between excitation of the sensor and response at resonance (e.g. 90°) to track the resonance frequencies. The slope of the phase curve at resonance, which can be identified via two different frequencies near resonance (e.g. at $90° \pm \alpha$) is related to the damping.

The focus of this work lies on the derivation of explicit, analytic equations that relate the vibrational parameters of the whole system (resonator + fluid) with the properties of the surrounding fluid for the case of low viscosity and elasticity.
Keywords: resonance sensor, vibration control, complex fluids, rheology, phase-locked loop.

1 Introduction

The measurement of the rheological properties of complex fluids is an important task in both scientific research and industrial applications. One of the major properties of a complex fluid is its viscoelastic behaviour, which describes the behaviour lying in between the classical extremes of a Newtonian fluid and

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Hookean elastic solid. Thus, viscoelastic materials have properties of both of these extremes, which leads to the fact that they exhibit time-dependent strain when a stress is applied. This behaviour is described by the relaxation time of a fluid, whereas a fluid can possess different relaxation times [1]. It can also be described by the complex coefficient of dynamic viscosity η^*, which is a generalization of the concept of viscosity for a viscoelastic fluid undergoing harmonic motion [2]. Since complex fluids exhibit different structural features that range over many orders of magnitude, their relaxation times span over a broad range of frequencies [1]. For this purpose, commercial rotating parallel-plate rheometers can vary their frequency continuously, hence providing a good frequency resolution. Unfortunately, due to the measurement principle, they have an upper frequency limit of about 100 Hz. With the aim to investigate fluids with short relaxation times, many instruments with the objective to operate at higher frequencies have therefore been developed in the past. Already in 1970, Schrag and Johnson [3] described the “multiple lumped resonator”, a torsionally vibrating rod consisting of several discrete masses that define five discrete resonance frequencies in the range of 100 Hz to 8300 Hz. The resonator is immersed in the sample fluid and excited at each of the resonance frequencies. The shift in resonance frequency and the damping form the base for the calculation of the real and imaginary part of η^*. A similar approach is followed by Stokich *et al.* [4], who uses eleven torsional vibration modes of a freely mounted quartz piezoelectric crystal. Whereas these devices are based on one resonator, Blom and Mellema [5] uses four differently dimensioned torsional pendulums to provide the frequency dependent information.

These instruments have all in common that they cover a broad frequency range and provide relatively good accuracy. However, most of them are rather for academic or laboratory use. The present work presents a device for the characterization of viscoelastic fluids that has a robust design and can thus be integrated in industrial processes. It provides continuous information of η^* and is therefore suitable as process or quality control, e.g. in chemical or food industry. The design is based on an existing viscosity sensor [2, 6], consisting of a torsionally vibrating hollow cylinder and has been extended to function at several vibration modes.

2 Measurement principle and sensor design

The sensor design which is used in this work is depicted in fig. 1. The vibrating part consists of an outer hollow cylinder (4), which is in contact with the sample fluid (7). It is at one end fixed to the body (1) and on the other side connected to an inner hollow cylinder (5), which leads inside the outer tube back into the body. At its free end a permanent magnet (3) is mounted, surrounded by a coil arrangement (2). The coils are placed such that an alternating current leads to a harmonic momentum, exciting the system to torsional vibrations. The arrangement of the tubes represents a continuous system, which possesses theoretically an infinite number of vibration modes i.e. resonance frequencies [7]. Due to the surrounding fluid, the resonance frequencies and the damping values of the resonator change. With a laser Doppler vibrometer (6) the velocity of the inner tube can be measured and used in a control loop, which controls the excitation frequency of the excitation moment. The laser beam is arranged as near as possible to the excitation to achieve nearly collocated control [8]. The control loop is based on a phase-locked loop (PLL), which measures the actual phase shift between excitation and laser signal and tries to reduce the difference between actual phase shift and the characteristic value at resonance ($\Delta\Theta_{res}$) by adjusting the excitation frequency. Therewith, one is able to find the resonance frequencies and maintain resonance excitation. Fig. 2 shows an exemplary phase shift vs. frequency curve for a one degree of freedom oscillator to illustrate the relationship between measured error $\Delta\Theta - \Delta\Theta_{res}$ and control variable ω.

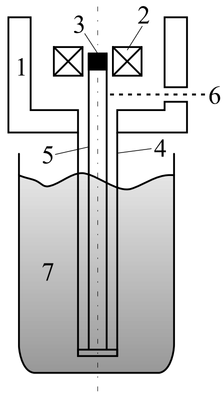

Figure 1: Illustration of the sensor design. (1) sensor body, (2) excitation coil, (3) magnet, (4) outer tube, (5) inner tube, (6) laser readout, (7) sample fluid.

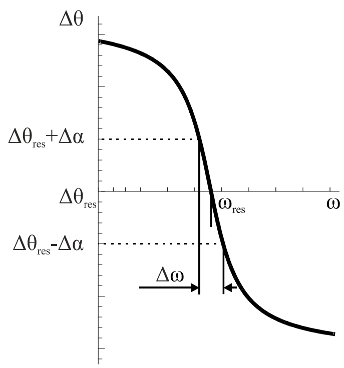

Figure 2: Exemplary phase spectrum between excitation moment and velocity readout near the resonance frequency.

The slope of the phase curve at resonance, which can be identified via two different frequencies near resonance (at $\Delta\Theta_{res} \pm \alpha$), is related to the damping coefficient δ

$$\delta = \Delta\omega \cdot \frac{1}{2\tan(\Delta\alpha)}. \tag{1}$$

3 Analytical model of the sensor

The derivation of analytical equations that connect the measurement parameters of the sensor with the properties of a viscoelastic fluid is presented in the following section. General non-linear viscoelasticity explores the dependence of the rheological properties of a fluid on the absolute strain it is subjected to [1]. Since the sensor performs a non-uniform, harmonic motion, the strain is not constant. Therefore, a linear viscoelastic fluid has to be assumed in the present context, which has a behaviour independent on the strain. It can thus be described with the frequency dependent complex viscosity

$$\eta^* = \eta'(\omega) - i\eta''(\omega) \tag{2}$$

which is similar to Newton's formulation, the ratio of shear stress τ to the rate of shear $\dot{\gamma}$ [1]

$$\tau(t) = \eta^*\dot{\gamma}(t). \tag{3}$$

To simplify the calculations and to provide useful equations at the end as well as to reach useful frequencies, the following specifications have been made:

a) The sensor shall be designed so that the lowest frequency without fluid is greater than 1000 Hz;
b) The material damping has to be low to increase the influence of the fluid;
c) The viscosity of the fluid is limited to $\eta' < 1$ Pas, the elasticity shall be smaller than the viscosity, $\eta'' \ll \eta'$.

3.1 A single rod with viscoelastic fluid loading

Since the interaction between fluid and structure occurs mainly on the outer tube, a torsionally vibrating, elastic rod is considered for the beginning, as depicted in fig. 3. Considering the rod without fluid, the differential equation can easily be set up with

$$GI_p\Phi''(z,t) - \rho I_p\ddot{\Phi}(z,t) = 0, \tag{4}$$

where Φ' denotes the derivative of the angular displacement Φ with respect to z and $\dot{\Phi}$ the derivative with respect to time *t* [7]. Using the separation ansatz

$$\Phi(z,t) = \Phi(t) \cdot \mathrm{e}^{i\frac{1}{\lambda_n}z}, \tag{5}$$

in which λ_n denotes the spatial wavelength according to the *n*-th vibration mode, the solution for $\Phi(t)$ can then easily be derived [7]. The resonance frequencies of the rod are given as

$$\omega_{res,n} = \sqrt{\frac{G}{\rho\lambda_n^2}}. \tag{6}$$

For the sake of readability the index *n* is omitted in the following calculations.

To model the material damping of the system the shear modulus can be written as a complex number $G = G_0\,(1 + i \cdot 1/Q)$, including the Q-factor of the material. The resonance frequency therewith becomes complex as well with

$$\omega_{res} = \sqrt{\frac{G_0}{\rho\lambda^2}\left(1 + i\,\frac{1}{Q}\right)} \approx \omega_{res,0} + i\,\frac{\omega_{res,0}}{2Q}, \tag{7}$$

including the undamped resonance frequency $\omega_{res,0}$ and the damping factor $\delta = \omega_{res,0}/2Q$. Following the calculation in [9] and introducing the complex viscosity (2), the shear stress of a viscoelastic, incompressible fluid at $r = R$ can be described by

$$\tau_{r\varphi} \approx \left(R\omega\sqrt{\frac{1}{2}\rho_f\omega\eta^*} \cdot (1 - i) - \frac{3}{2}i\omega\eta^*\right) \cdot \Phi(z,t) = \hat{\tau}_{r\varphi}\Phi(z,t) \tag{8}$$

where ρ_f denotes the fluid density. The shear stress exerts an external moment on the rod, which changes eqn. (4) to

$$G\Phi''(z,t) - \rho\ddot{\Phi}(z,t) = \kappa\hat{\tau}_{r\varphi}\Phi(z,t)\,, \tag{9}$$

$$\kappa = \frac{2\pi R^2}{I_p} = \frac{4R^2}{(R^4 - r_i^4)}. \tag{10}$$

Using again ansatz (5), eqn. (9) simplifies to the homogeneous differential equation

$$\ddot{\Phi}(t) + \left(\frac{G}{\rho\lambda^2} - \frac{\kappa}{\rho}\left(\Re\{\hat{\tau}_{r\varphi}\} + i\Im\{\hat{\tau}_{r\varphi}\}\right)\right)\Phi(t) = 0 \tag{11}$$

in which $\hat{\tau}_{r\varphi}$ has been split up in its real part $\Re\{\hat{\tau}_{r\varphi}\}$ and imaginary part $\Im\{\hat{\tau}_{r\varphi}\}$.

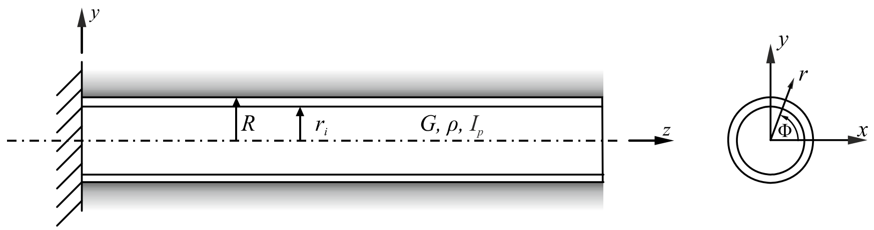

Figure 3: Single infinite rod, supported at one end and surrounded by fluid. *G* presents the shear modulus, ρ the density of the material, I_p denotes the polar moment of inertia.

With the assumption that there exist no reciprocal effects between frequency dependent fluid forces and oscillation frequency, the resonance frequency can be directly extracted from eqn. (8) with (6) as

$$\omega_{res}^{f} = \sqrt{\omega_{res} - \frac{\kappa}{\rho}\left(\Re\{\hat{\tau}_{r\varphi}\} + i\Im\{\hat{\tau}_{r\varphi}\}\right)} \approx \omega_{res} - \frac{1}{2}\frac{1}{\omega_{res}}\frac{\kappa}{\rho}\left(\Re\{\hat{\tau}_{r\varphi}\} + i\Im\{\hat{\tau}_{r\varphi}\}\right) \quad (12)$$

The shift in frequency and the additional damping can then be written as

$$\Delta\omega = \omega_{res,0} - \omega_{res,0}^{f} \approx \frac{1}{2}\frac{1}{\omega_{res,0}}\frac{\kappa}{\rho}\Re\{\hat{\tau}_{r\varphi}\}, \quad (13)$$

$$\Delta\delta = \delta^{f} - \delta \approx \frac{1}{2}\frac{1}{\omega_{res,0}}\frac{\kappa}{\rho}\Im\{\hat{\tau}_{r\varphi}\}, \quad (14)$$

A series expansion around $\eta'' = 0$ can be applied to simplify the real and imaginary part of $\hat{\tau}_{r\varphi}$ in eqn. (13) and (14). Taking the squared value, the easy formulations

$$\Delta\omega^{2} = \frac{1}{8}\kappa^{2}\frac{R^{2}}{\rho^{2}}\rho_{f}\omega_{res,0}\cdot(\eta' - \eta'') \quad (15)$$

$$\Delta\delta^{2} = \frac{1}{8}\kappa^{2}\frac{R^{2}}{\rho^{2}}\rho_{f}\omega_{res,0}\cdot(\eta' + \eta'') \quad (16)$$

are finally obtained. Hence, the viscosity decreases the resonance frequency and increases the damping, whereas the elastic part increases both, the damping and the frequency. These results are plausible, considering the fluid as an additional mass, spring and damping element.

3.2 Sensor equations for the whole resonator

Since the complete resonator consists of several parts (fig. 4), which are only partially surrounded by the fluid, the sensor equations will become more complex. It is thus not possible to derive analytic equations as it was done in section 3.1. However, neglecting effects at the tip of the rod, it can be assumed that the fluids influence on the whole sensor will be, analogue to eqns (15) and (16), of the form

$$\begin{aligned}\Delta\omega_{n}^{2} &= \boldsymbol{a_n}\cdot\rho_{f}\omega_{res,n,0}\cdot(\eta' - \eta'')\\ \Delta\delta_{n}^{2} &= \boldsymbol{a_n}\cdot\rho_{f}\omega_{res,n,0}\cdot(\eta' + \eta'')\end{aligned} \quad (17)$$

where the factor a_n is assumed to be constant but dependent on the mode shape. As previously introduced, the index *n* denotes the number of the vibration mode. To prove this assumption an impedance based semi-analytic model has been derived to describe the behaviour of the whole system. Thereby, a system can be separated in different elements, whose mechanical impedances Z_i can be described with standard formulations. The whole system is then described by a series connection of these impedances. From [9], the impedance Z_i of a fluid loaded tube like in fig. 3 is given at $z = l$ as

$$Z_i(l) = \frac{\cos(k'l) + \frac{ik'}{\omega}\frac{GI_p}{Z_{i-1}}\sin(k'l)}{\cos(k'l) + \frac{i\omega}{k'}\frac{Z_{i-1}}{GI_p}\sin(k'l)}, \qquad k'^2 = \frac{\omega^2\rho}{G} + \frac{\kappa}{G}\hat{\tau}_{r\varphi} \tag{18}$$

where Z_{i-1} denotes the impedance of the adjacent element at $x = 0$. With this method, the complex impedance at the magnet position can be formulated. This formulation corresponds to the complex ratio of the exciting moment and the angular velocity at the excitation position [9]. The resonance frequency and damping value can then be found numerically.

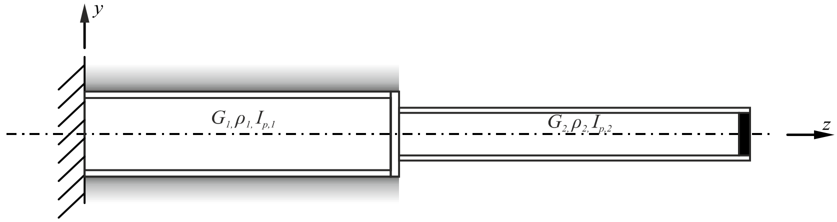

Figure 4: Physically equivalent representation of the resonator, consisting of inner tube, outer tube, connecting piece and magnet. The clamping on the left represents the sensor body.

The result of this computation is depicted in fig. 5. It shows the semi-analytic result of $\Delta\omega_1^2$ as a function of η' and η'' (black dots). The resonance frequency without fluid is $f_{res,1,0} = 5300$ Hz, the material damping $Q = 4000$. The surface is generated by a linear fit based on eqn. (17a) that is computed with MATLAB using the Levenberg-Marquard algorithm. It can be seen that the presumption that has been made at the beginning of this chapter holds very well for the specified viscosity and elasticity range. The relative error is below 2% for $\eta' > 5\eta''$, the coefficient of determination of the fit is $R^2 = 0.998$. The same result is obtained for the other frequencies, as expected with different factor $\boldsymbol{a}$, since the participation of the fluid loaded part changes.

One can thus state that the results for a single rod can be generalized for systems composed of several torsionally vibrating cylinders with partially viscoelastic fluid loading. However, the proportionality factor of this general form is not known a priori and has to be determined by other means.

Finally, knowing the constants $\boldsymbol{a_i}$, eqn. (17) can be transformed so that a direct measurement of η^* is possible via

$$\eta'(\omega_{res,n}) = \frac{1}{2a_n}(\Delta\delta_n^2 + \Delta\omega_n^2) \tag{19}$$

$$\eta''(\omega_{res,n}) = \frac{1}{2a_n}(\Delta\delta_n^2 - \Delta\omega_n^2)$$

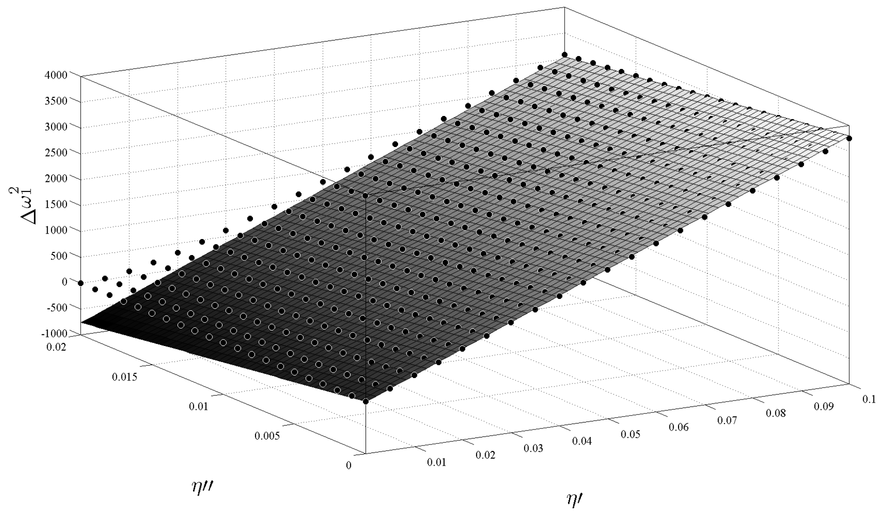

Figure 5: Squared value of the shift in resonance frequency in dependence on η' and η''. The points are analytical results from the impedance model, the surface is generated by a linear fit of the form $a \cdot (\eta' - \eta'')$, according to eqn. (17).

4 Summary and conclusion

The method presented here allows a practical and easy continuous measurement of the viscoelastic parameters of complex fluids. In consequence of the design the material exerts continuous vibrations. Since the measurement principle enables the use of several vibration modes, one is thus able to use several frequencies for the measurement. Therefore the sensor is able to detect the frequency dependence of the fluid parameters, but with the limitation to the resonance frequencies. On the other hand, the occurrence of continuous vibrations leads to very complex sensor equations that cannot be solved explicitly.

In this work it has been shown that it is possible to reduce the analytical description of the influence of a viscoelastic solid on a combined torsional rod structure to easy linear equations, which allows to directly extract η^* from the shift in resonance frequency and change in damping

These equations contain only one unknown parameter, which is dependent on the material and geometric properties of the structure. It can be described as a sensor constant that has to be found before using the sensor. Due to the fact that only one constant is needed per vibration mode, the determination of this constant can be executed by a calibration with fluids of either constant viscosity or elasticity. Newtonian calibration fluids ($\eta'' = 0$) are therefore well suited to provide the sensor constants. First experiments have shown that these theoretical results are valid for Newtonian Fluids. Further experiments have to be conducted with viscoelastic fluids.

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