Nonstationary problem moisture elasticity for a nonhomogeneous hollow thick-walled cylinder

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Abstract

This paper contains a solution of the problem of determining stress state in soil near a cylindrical cavity by the propagation of the moisture out of the cavity into the solid mass. The problem of moisture transfer is solved analytically in the nonstationary symmetric formulation. A result of solving this problem is the distribution of moisture in the array at any time until the release of a steady state. Given the slow diffusion of water from the hole to the periphery the problem of determining the stress state is solved in a quasi-stationary formulation. A feature of the calculation is based accounting depending soil deformation module of moisture, which is especially characteristic for these species as clay and loess soils. For example, for the elastic modulus of clay when the relative humidity changes from 0.3 to 1 can vary by more than 10 times. Thus our problem is to be solved by the methods of the theory of elasticity of inhomogeneous bodies.The problem is reduced to a differential equation with variable coefficients. This complicates the solution of the problem compared with the solutions for constant modulus of elasticity, but it provides a more accurate solution.
Keywords: heat and mass transfer, moisture elasticity, inhomogeneity, nonstationary problem, ground, clay, thick-walled shell.

1 Introduction

With bursting of pipework's, flooding and other emergency occurs the spread of moisture in the soil mass, which leads to appearance of forced deformations. which leads to swelling of soils, and the stress of the soil to failures and the collapse of buildings. This results in bulking of soil, the appearance stress-state in the array, which can cause failures of the soil and the collapse of buildings.

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These phenomena are characteristic of clay soils, loess, etc. Thus, the problem of determining the stress-strain state under the influence of moisture is important. There are some solutions of classical problems of moisture elasticity for homogeneous bodies (e.g. Ter-Martirosian [1] and others]. However, as shown by experimental studies, such as [2, 3], with moisture change the deformation characteristics of soils (especially modulus of deformation). However, experimental studies, such as [2, 3], show that with moisture the deformation characteristics of soils (especially modulus of deformation) change. Uneven moisture distribution makes inhomogeneous array, and we have to solve problems in the mechanics of inhomogeneous bodies. In (authors [4, 5]) obtained some sample problems moisture elasticity in stationary setting in view of dependence of the deformation modulus on humidity. It is shown that the stress state of the inhomogeneous array is significantly different from what occurs in the calculation for constant modulus of deformation.

2 Nonstationary problem of moisture transfer in thick-walled tube

Nonstationary process of moisture transfer described by the second Fick's law:

$$\frac{\partial w}{\partial t} = c_w \nabla^2 w \tag{1}$$

where c_w - coefficient of moisture conductivity, which is calculated by the formula [1]:

$$c_w = \frac{k_w \alpha (1+e)}{\gamma_s}.$$

Here $k_w = k_f (w/w_s)^i$ - the filtration coefficient of partial flow; k_f – filtration coefficient of continuous flow; w_s – the moisture of saturated soil; i – relative ice content; α – coefficient of proportionality; $e = n/(1-n)$- the coefficient of soil porosity; n – porosity of the soil; γ_s - the density of the soil particles.

We consider a cylindrical array of inner radius a, and outer – b = 10a. The problem is solved with the following initial and boundary conditions:

$$w(r,0) = w_b\,, \tag{2}$$

$$w(a,t) = w_a = \boldsymbol{\kappa}\, w_b\,,$$

$$w(b,t) = w_b\,,$$

where $\boldsymbol{\kappa} \geq 0$. When $\boldsymbol{\kappa} < 1$ will be drying out from inside of the cylinder. When $\boldsymbol{\kappa} = 1$ would be an equilibrium state. We are interested in the case when $\boldsymbol{\kappa} > 1$, as he describes the process of moistening from inside of the cylinder, such as rupture of the pipe.

Eqn (1) taking into account axial symmetry takes the form:

$$\frac{\partial w}{\partial t} = c_w \left(\frac{d^2 w}{dr^2} + \frac{1}{r} \frac{dw}{dr} \right). \tag{3}$$

To solve the problem in dimensionless form, we use the following substitutions:

$$t = \frac{a^2}{c_w} \tau, \quad r = a\rho. \tag{4}$$

Eqn (3) with (4) becomes:

$$\frac{\partial w}{\partial \tau} = \frac{\partial^2 w}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial w}{\partial \rho} \tag{5}$$

The solution of eqn (5) will be sought in the form [6]:

$$w(\rho, \tau) = \theta(\rho) + \vartheta(\rho, \tau), \tag{6}$$

where $\theta(\rho)$ satisfies to equation:

$$\frac{d^2 \theta}{d\rho^2} + \frac{1}{\rho} \frac{d\theta}{d\rho} = 0 \tag{7}$$

and next boundary conditions:

$$\theta(\rho_a) = w_a, \quad \theta(\rho_b) = w_b. \tag{8}$$

Here $\rho_a = 1; \rho_b = b/a$.

The function $\vartheta(\rho, \tau)$ satisfied to equation:

$$\frac{\partial \vartheta}{\partial \tau} = \frac{\partial^2 \vartheta}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \vartheta}{\partial \rho} \tag{9}$$

and the following initial and boundary conditions:

$$\begin{gathered} \vartheta(\rho, 0) = w_b, \\ \vartheta(\rho_a, \tau) = w(\rho_a, \tau) - \theta(\rho_a) = 0, \\ \vartheta(\rho_b, \tau) = w(\rho_b, \tau) - \theta(\rho_b) = 0 \end{gathered} \tag{10}$$

In this case, a replacement (6) will be identical to the desired function.

The solution eqn (7) in mind (8) is:

$$\theta(\rho) = \frac{w_b \ln(\rho/\rho_a) - w_a \ln(\rho/\rho_b)}{\ln(\rho_b/\rho_a)} \tag{11}$$

To determine the function $\vartheta(r,t)$ use the method of separation of variables, presenting this function as a product:

$$\vartheta(\rho,\tau) = R(\rho)\cdot T(\tau) \tag{12}$$

Substituting (12) into (10) we get

$$R\cdot\partial T/\partial\tau = \partial^2 R/\partial\rho^2\cdot T + \partial R/\partial\rho\cdot T/\rho.$$

After simple transformations, we obtain the equation:

$$\frac{1}{T}\frac{\partial T/\partial\tau}{} = \frac{\partial^2 R/\partial\rho^2 + (\partial R/\partial\rho)/\rho}{R}.$$

Left-hand side depends only on τ, and the right only on ρ. Equality will be true only if both sides do not depend on τ, nor on ρ, i.e.:

$$\frac{1}{T}\frac{\partial T}{\partial\tau} = \frac{1}{R}\left(\frac{\partial^2 R}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial R}{\partial\rho}\right) = A = const$$

A constant A replace by expression $-k^2$ to show that it is negative. If it is positive, then the humidity would increase indefinitely with time, that contrary to the process under consideration.

Integrating both of the equality, we obtain expressions for functions T and R:

$$T(\tau) = B\cdot\exp(-k^2\tau),$$

$$R(\rho) = C_1 J_0(k\rho) + C_2 Y_0(k\rho).$$

Here J_0 and Y_0 respectively Bessel functions of first and second kind zero order.

As a result, using (12) we obtain

$$\vartheta(\rho,\tau) = [C_1 J_0(k\rho) + C_2 Y_0(k\rho)]\exp(-k^2\tau). \tag{13}$$

From the boundary conditions (10) follow the equalities

$$\begin{cases} C_1 J_0(k\rho_a) + C_2 Y_0(k\rho_a) = 0, \\ C_1 J_0(k\rho_b) + C_2 Y_0(k\rho_b) = 0. \end{cases} \tag{14}$$

The system (14) will have a nontrivial solution if its determinant is equal to 0:

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$$\begin{vmatrix} J_0(k\rho_a) & Y_0(k\rho_a) \\ J_0(k\rho_b) & Y_0(k\rho_b) \end{vmatrix} = 0,$$

or

$$J_0(k\rho_a)Y_0(k\rho_b) - J_0(k\rho_b)Y_0(k\rho_a) = 0. \tag{15}$$

Bessel functions are aperiodic and have infinite number of roots. Therefore, the function $\vartheta(\rho,\tau)$ will be a fundamental system of solutions, and the eigenvalues of this system can be found from (15). As a result function $\vartheta(\rho,\tau)$ takes the form:

$$\vartheta(\rho,\tau) = \sum_{n=1}^{\infty}\left[C_1 J_0(k_n\rho) + C_2 Y_0(k_n\rho)\right]\exp(-k_n{}^2\tau).$$

Substituting this equality and (11) in (6), we obtain:

$$w(\rho,\tau) = \frac{w_b \ln(\rho/\rho_a) - w_a \ln(\rho/\rho_b)}{\ln(\rho_b/\rho_a)} + \sum_{n=1}^{\infty}\left[C_1 J_0(k_n\rho) + C_2 Y_0(k_n\rho)\right]\exp(-k_n{}^2\tau)\cdot$$

Expressing by the first of (14) C_2 by C_1, and using the properties of Bessel functions [7], we write a general expression for the function $w(\rho,\tau)$:

$$w(r,t) = \frac{w_b \ln(\rho/\rho_a) - w_a \ln(\rho/\rho_b)}{\ln(\rho_b/\rho_a)} + \sum_{n=1}^{\infty}\frac{V_0(k_n\rho)\exp(-k_n{}^2\tau)}{J_0^2(k_n\rho_a) + Y_0^2(k_n\rho_b)} \times$$
$$\times\left\{ \frac{\pi^2 k_n^2}{2} J_0(k_n\rho_b)\int_{\rho_a}^{\rho_b}\rho f(\rho) V_0(k_n\rho)\,d\rho - \right.$$
$$\left. - \pi J_0(k_n\rho_b)\left[w_b J_0(k_n\rho_a) - w_a Y_0(k_n\rho_b)\right] \right\},$$

where $V_0(k_n\rho) = J_0(k_n\rho)Y_0(k_n\rho_a) - J_0(k_n\rho_a)Y_0(k_n\rho)$, $f(\rho)$ – function of the initial moisture distribution. Since in our case $f(\rho) = w_b = const$, then the function moisture assumes the form:

$$w(\rho,\tau) = \frac{w_b \ln(\rho/\rho_a) - w_a \ln(\rho/\rho_b)}{\ln(\rho_b/\rho_a)} + \pi(w_a - w_b)\sum_{n=1}^{\infty}\exp(-k_n{}^2\tau) \times$$
$$\times J_0^2(k_n\rho_b)\frac{J_0(k_n\rho)Y_0(k_n\rho_a) - J_0(k_n\rho_a)Y_0(k_n\rho)}{J_0^2(k_n\rho_a) + J_0^2(k_n\rho_b)}. \tag{16}$$

In the software package Matlab R2010b was simulated moisture distribution along the radius at different times. The interval from ρ_a to ρ_b was divided into 100 steps. Since the exact solution consists of an infinite number terms of series the Bessel functions, we can show only an approximate solution with a limited number of terms. Fig. 1 shows the distribution of moisture along the radius at

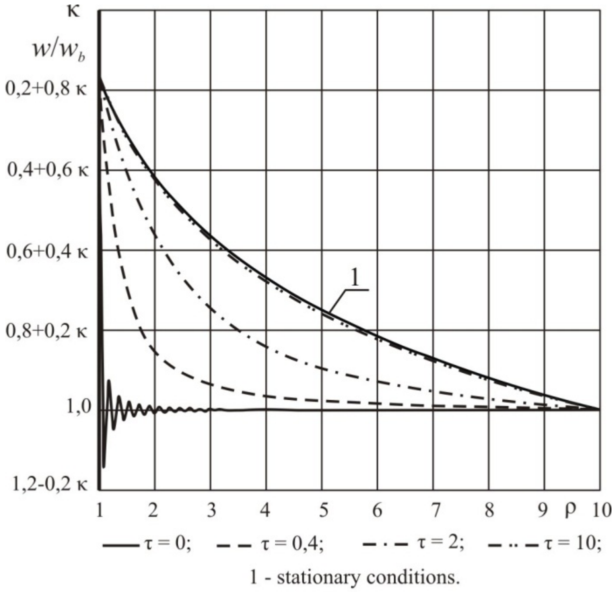

Figure 1: Moisture distribution along the radius at different times.

different times when the number of terms of series N = 100. We recall that $\kappa = w_a / w_b$ (see (2)). When τ = 10 moisture distribution is installed and almost coincides with the solution of the corresponding stationary problem [5].

3 The calculation of the stress state

Because the speed of propagation of the field moisture is significantly less than the speed of propagation of elastic waves the medium under consideration, and the generating factor of stress-stain state is the only field moisture, it can be assumed that the particles of the medium at a time move instantaneously. It means that to derivate resolving equation you can use the static equation, and the time (t or τ) will be part of it as a parameter. Resolving equation axisymmetric problem of the theory for inhomogeneous bodies in cylindrical coordinates (plane strain state) in the presence of forced strain ε_w for the case when the Poisson's ratio ν = const, is [8]:

$$\frac{d^2\sigma_r}{d\rho^2} + \left(\frac{3}{\rho} - \frac{E'}{E}\right)\frac{d\sigma_r}{d\rho} - \frac{E'}{E}\frac{1-2\nu}{1-\nu}\frac{\sigma_r}{\rho} = -\frac{E}{\rho}\frac{1}{1-\nu}\varepsilon_{\mathrm{w}}' \tag{17}$$

In eqn (17) and then the prime denotes the partial derivative with respect to dimensionless radius ρ. In this problem, the swelling strain, at a constant value the coefficient of linear expansion β_{sw} equal to $\varepsilon_w = \beta_{sw}\Delta w$, where

$$\Delta w(\rho,\tau) = w(\rho,\tau) - w(\rho,0).$$

Substituting this in (17), we obtain:

$$\frac{d^2\sigma_r}{d\rho^2} + \left(\frac{3}{\rho} - \frac{E'}{E}\right)\frac{d\sigma_r}{d\rho} - \frac{E'}{E}\frac{1-2\nu}{1-\nu}\frac{\sigma_r}{\rho} = -\frac{E}{\rho}\frac{1}{1-\nu}\beta_{sw}\frac{\partial(\Delta w)}{\partial\rho}, \qquad (18)$$

Assuming approximately that pressure of the soil mass on considered thick-walled cylinder is constant and equal to γH, where H – the depth of the hole, we can write the boundary conditions for the stresses

$$\sigma_r(\rho_a) = 0; \qquad \sigma_r(\rho_b) = -\gamma H .$$

For example, we consider material as clay. It is characterized by the following parameters [1, 9]: $k_w = k_f = 5\cdot10^{-7}$m/s; w_s =0.363; i = 0; n=0.5; e = 1; α = 6.468kN/m^2; γ_s = 26.95kN/m^3. Hence, we obtain c_w = 2.4·10^{-7}m^2/s. Below is a sample calculation for the following input data: a = 0.25 m; b = 2.5 m; $w_a = w_s = 0.363$; $w_b = w(\tau=0) = 0.2$.

Taking H = 10 m, we find that γH = 0.2695 MPa. As in the problems considered in [4, 5], Poisson's ratio equal ν = 0.4, and the modulus of elasticity, which introduces inhomogeneity in problem is $E = 19{,}88(w/w_s)^{-2{,}4}$ (approximation of the experimental data presented in [4, 5]). Appearing in (18) relations E'/E and $\partial(\Delta w)/\partial\rho$ are:

$$\frac{E'}{E} = -2{,}4\frac{w'}{w}; \qquad \frac{\partial(\Delta w)}{\partial\rho} = w'.$$

In these equalities

$$w' = \frac{\partial w}{\partial\rho} = \frac{w_b - w_a}{\rho\ln(\rho_b/\rho_a)} + \pi(w_a - w_b)\sum_{n=1}^{\infty}\exp(-k_n^2\rho)\times$$
$$\times J_0^2(k_n\rho_b)k_n\frac{J_0(k_n\rho_a)Y_0(k_n\rho) - J_0(k_n\rho)Y_0(k_n\rho_a)}{J_0^2(k_n\rho_a) - J_0^2(k_n\rho_b)}.$$

Tangential normal stresses are known in the theory of elasticity relations for axisymmetric problems:

$$\sigma_\theta = \sigma_r + r\cdot\sigma_r'$$

In Figs 2 and 3 show the graphs of the stress distribution along the radius at different times at N = 100. Just like in considering in the first part the problem of

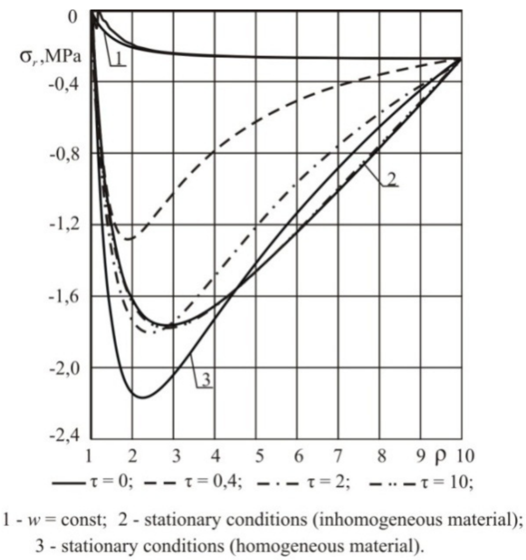

Figure 2: The distribution stress σ_r along the radius at different times.

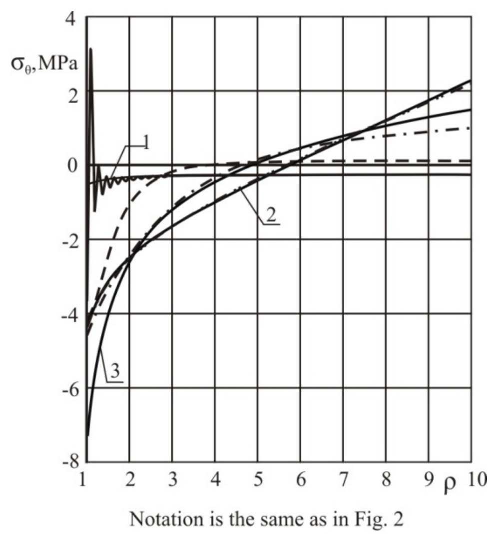

Figure 3: The distribution stress σ_θ along the radius at different times.

the distribution of moisture at time τ = 10 stress is almost the same as in the steady state. The figures show also the solution of the homogeneous problem with modulus equal to:

$$E_{\text{av}} = \frac{\int_{\rho_a}^{\rho_b} E(w(\rho))d\rho}{\rho_b - \rho_a} = 52{,}27\text{MPa}.$$

For the validation of the results we'll do static test which consists in the balance half of the cylinder. From this test, it follows that

$$\int_{\rho_a}^{\rho_b} \sigma_\theta(\rho)d\rho = \rho_a\,\sigma_r(\rho_a) + \rho_b\sigma_r(\rho_b).$$

In our case $\sigma_r(\rho_a) = 0$ and $\sigma_r(\rho_b) = -\gamma H.$ Right side of equality (19) is equal (–2.695MPa), and the value of the integral on the left-hand side of (19), calculated with N= 100 for various volumes are given in Table 1.

Table 1: Left side of equality (19).

τ	$\int_{\rho_a}^{\rho_b} \sigma_\theta(\rho)d\rho$
0	-2,695
0,4	-2,641
2	-2,692
10	-2,690
Stationary conditions	-2,697

4 Conclusions

According to Fig. 1 can be seen that the greatest difference of exact and approximate solutions is observed at the initial time, when the contribution to the solution of a series of Bessel functions is greatest. In these moments of time at low values of the radius moisture distribution is rather sharp character, which driven by a jump function on the boundary at $\rho = 1$. Also from Fig. 1 implies that when $\tau = 10$ moisture distribution is almost established mode. According to Figs 2 and 3 we see that for large time ($\tau = 10$), thestresses is almost equal to the solution obtained for stationary conditions. When performing a static test the maximum error is 2% ($\tau = 0.4$) that indicates a sufficiently high precision calculations. Especially note the significant impact on the results inhomogeneity due to the dependence of the elastic modulus on moisture. For example, for the stress σ_θ difference is about 40%.

We conclude that the inhomogeneity of the materials due to moisture is very important not only in soil mechanics, but also in the calculation of concrete, wood and other structures that are in a wet environment.

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