STOKES EQUATION SOLUTION USING THE LOCALIZED METHOD OF FUNDAMENTAL SOLUTIONS WITH A GLOBAL BASIS

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ABSTRACT
The paper focuses on deriving a local variant of the method of fundamental solutions (MFS) for the case of Stokes flow. Compared to the global and local basis variants, the local with global basis one leads to a sparse characteristic matrix as in fully localized variants but with a narrower system of equations and thus makes the solution of especially large-scale problems more efficient. It is also essential to keep the condition number of the characteristic matrix within reasonable bounds and remove the solution dependency on fictitious sources. A combination of MFS and finite collocation approach was used for the localization with a globally defined Stokeslet fundamental solution. The results of the particular local variant were compared on several examples, and the dependence of the solution on the density of the point network and the dimensions of the stencil used were also tested in the paper.
Keywords: method of fundamental solutions, biharmonic equation, Stoke's flow.

1 INTRODUCTION

The linear case within the broader Navier–Stokes equations framework, which describes fluid motion, is called Stokes flow. In the context of former conditions, one can assume that the fluid is considered incompressible, and the flow velocity is very slow.

There are various approaches to solving the Stokes flow problem numerically. One such approach, the vorticity-stream function formulation, leverages the relationship in two dimensions between vorticity and the Laplacian of the stream function [1]. By eliminating vorticity, one obtains the biharmonic equation for the stream function, and the Stokes flow is solved in the form of the biharmonic equation.

Unlike traditional methods, reliant on predefined meshes, meshless numerical methods use scattered data points to approximate solutions in a continuous domain. These methods have gained popularity due to their effectiveness in handling complex geometries, adaptive refinement, and straightforward implementation. Over the past two decades, several meshless methods have emerged, including the Trefftz-like approaches represented by the method of fundamental solutions (MFS) [2]–[5], singular boundary method (SBM) [6] and boundary knots method (BKM) [7]. SBM, like MFS, employs the PDE's fundamental solution as an interpolation function, but it faces challenges related to singularities in the interpolation matrix. Conversely, the boundary knot method uses the general PDE solution as an interpolation function, resulting in regular diagonal terms. However, when dealing with a general PDE solution, the characteristic matrix may encounter issues related to its ill-conditioned character. Finding suitable general solutions for Laplace and biharmonic equations can be complex, and finding the proper technique to evaluate origin intensity factors in the case of SBM is also challenging. One of the possibilities for overcoming the mentioned issues is to use the fundamental solutions defined globally with the singular sources placed outside the application domain as in MFS and use them as a base for all local domains.

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Recent attention has turned towards localized variations of the Trefftz-like method, aiming to enhance matrix conditioning. These variants adopt different localization techniques, primarily centred on applying the appropriate method within limited neighbourhoods of specific points. The side effect of the localized technique is the lowering of the fundamental solution source position influence on the final solution.

The MFS localized forms [3], [4] adopt mainly the local subdomain concept using the internal nodes with the fictitious boundary to evaluate the solution in the solution centre; the present method forms the local model using the convex hull of the local domain boundary nodes and the global sources divided regularly around the global domain. On the other hand, a localized numerical solution governed by a biharmonic equation (stream function form of the Stokes equation) is somewhat tricky because of the need to impose the two boundary conditions. A biharmonic problem is solved using the localized method of fundamental solutions presented in Fan et al. [5]. However, this formulation [5] uses the concept of a support domain with internal nodes and the problematic part caused by the imposition of the Neumann boundary conditions results in the overdetermined linear system with velocities in the direction of local model domains – the present formulation results in the regular linear system with the velocities as the part of the solution.

The localization principle is universally applicable, especially for methods emphasizing boundary points like the singular boundary method (SBM) or method of fundamental solutions (MFS). This principle defines a local solution region based solely on boundary points [8]. It mainly benefits the application of MFS, leading to the development of the local method of fundamental solutions (LMFS). In LMFS, MFS is used for local PDE solutions within overlapping subdomains of interior points, and a global sparse system of linear equations is assembled to determine the unknown values of the area.

The initial sections of this article present the description of the LMFS principle, along with their application to solving the two-dimensional Stokes flow problem. Subsequent sections present the results of solving the lid-driven cavity problem.

2 BASIC FLOW EQUATIONS

When the influence of inertial forces is much weaker than viscous forces and pressure gradient, one can label the flow as Stokes flow. Then, it is possible to describe 2D Stokes flow using the momentum (1) and continuity (2) as follows:

$$\frac{\partial p}{\partial x_i} - \Delta u_i = 0, \tag{1}$$

$$\frac{\partial u_i}{\partial x_i} = 0. \tag{2}$$

In these equations, u_i stands for the ith velocity component, and p represents pressure. Expressing the momentum and continuity equations relying on the concepts of vorticity ω and stream function Ψ (for more detail, see Fan et al. [9]) provides an alternative formulation of the Stokes equations as follows:

$$\Delta\omega = 0, \tag{3}$$

$$\Delta\Psi = -\omega. \tag{4}$$

Cancelling the vorticity term from eqn (4) results in the biharmonic form of the Stokes equation.

$$\Delta^2\Psi = 0. \tag{5}$$

For boundary value problems governed by biharmonic PDE, the two types of boundary conditions should be specified simultaneously along the boundary Γ.

$$\Psi(\mathbf{x}) = 0 \qquad \mathbf{x} \in \Gamma, \tag{6}$$

$$\frac{\partial \Psi(\mathbf{x})}{\partial \mathrm{x}_i} = (-1)^{(i+1)} u_{i0}. \tag{7}$$

In this context, u_{i0} represents the prescribed velocity components at the boundary Γ.

3 IMPLEMENTATION OF THE LOCALIZED MFS

The study aimed to solve eqn (5) while considering boundary conditions (6) and (7) within a two-dimensional domain Ω and its boundary Γ. We achieved this by creating a set of internal and boundary points that covered the entire global computational domain [10]. Near each internal node, we defined a small local domain (see Fig. 1). Within these domains, we applied the method of the fundamental solutions to solve the biharmonic equation, considering the unknown values at the local domain boundary, which are marked as 'solution centres'. The specified global boundary condition is applied if a solution centre is on the global boundary (6) and (7).

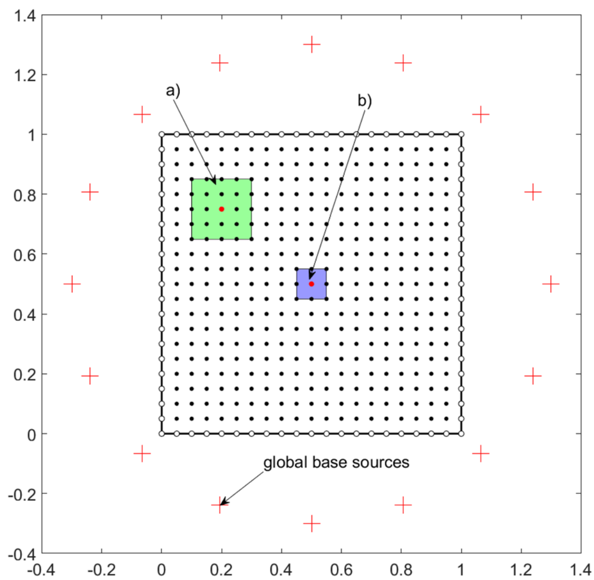

Figure 1: Regular local subdomains. (a) Eight solution centres; and (b) Sixteen solution centres on the local boundary.

For solving eqn (5) within each local domain Ω Ωk, we utilized the method of the fundamental solutions (MFS). In the evaluation of the stream function ψ and $\partial\psi/\partial\mathbf{n}$, we employed fundamental solutions of Laplace and the biharmonic operator [11] in the form:

$$\Psi(\mathbf{x}_i) = \sum_{j=1}^{n} G_{\mathrm{B}}(r_{ij})\alpha_j + G_L(r_{ij})\beta_j, \tag{8}$$

$$\frac{\partial \Psi(\mathbf{x}_i)}{\partial \mathbf{n}} = \sum_{j=1}^{n} \frac{\partial G_{\mathrm{B}}(r_{ij})}{\partial \mathbf{n}} \alpha_j + \frac{\partial G_L(r_{ij})}{\partial \mathbf{n}} \beta_j. \tag{9}$$

In these equations, n is the number of solution centres, and G_L and G_B were the fundamental solutions for Laplace and the biharmonic equation, respectively, which are defined as:

$$G_B(r_{ij}) = -\frac{1}{8\pi} r_{ij}^2 \ln(r_{ij}), \tag{10}$$

$$G_L(r_{ij}) = -\frac{1}{2\pi} \ln(r_{ij}), \tag{11}$$

where $r_{ij} = \|\mathbf{x}_i - \mathbf{x}_i^f\|$. To formulate the MFS solution within the local domain, we set up a system of linear equations:

$$\sum_{j=1}^{2n} A_{ij} a_j = d_i, i = 1, \dots, 2n. \tag{12}$$

This system's structure depended on the local domain's number of solution centres n. Matrix **A** and vectors **a** and **d** took the following form:

$$\mathbf{A} = \begin{bmatrix} G_B(r_{cj}) & G_L(r_{cj}) \\ \frac{\partial G_B(r_{cj})}{\partial \mathbf{n}} & \frac{\partial G_L(r_{cj})}{\partial \mathbf{n}} \end{bmatrix}; \ \mathbf{a} = \begin{Bmatrix} \alpha_j \\ \beta_j \end{Bmatrix}; \ \mathbf{d} = \begin{Bmatrix} \Psi_j \\ \frac{\partial \Psi_j}{\partial \mathbf{n}} \end{Bmatrix}. \tag{13}$$

The values a_j in eqn (12) represented interpolation coefficients α and β for the given local domain. The vector d_i on the right side of eqn (12) contained the unknown values of Ψ and $\partial\Psi/\partial\mathbf{n}$ at the solution centres on the local domain's boundary. The values of Ψ and $\partial\Psi/\partial\mathbf{n}$ at the local domain's central point could be evaluated as follows:

$$\Psi_i(\mathbf{x}_c) = \sum_{j=1}^{n} G_B(r_{cj})\alpha_j + G_L(r_{cj})\beta_j, \tag{14}$$

$$\frac{\partial \Psi_i(\mathbf{x}_c)}{\partial \mathbf{n}} = \sum_{j=1}^{n} \frac{\partial G_B(r_{cj})}{\partial \mathbf{n}} \alpha_j + \frac{\partial G_L(r_{cj})}{\partial \mathbf{n}} \beta_j. \tag{15}$$

Here, r_{jc} is the radial distance between the central point x_c and solution centre j. Eqn (12) could be expressed in matrix notation as:

$$\mathbf{d} = \mathbf{G}_c \mathbf{a}. \tag{16}$$

From eqn (12), we could express the vector **a** as:

$$\mathbf{a} = [\mathbf{A}]^{-1} \mathbf{d}. \tag{17}$$

If we substituted this expression into eqn (13), we obtained:

$$\mathbf{b} = \mathbf{G}_c [\mathbf{A}]^{-1} \mathbf{d} = \mathbf{W}_c \mathbf{d}. \tag{18}$$

In this equation, $\mathbf{W}_c$ represented a stencil weight vector [12] that could be used to construct the global system of equations. This system, characterized by sparsity, could be defined as:

$$\Psi_k - \sum_{j=1}^{n} W_j^k \Psi_j - \sum_{j=n+1}^{2n} W_j^k \frac{\partial \Psi_j}{\partial x_i} n_{ij} = \sum_{l=1}^{m} W_l^k \Psi_{0l}, \tag{19}$$

$$\frac{\partial \Psi_k}{\partial x_i} - \sum_{j=1}^{n} \frac{\partial W_j^k}{\partial x_i} \Psi_j - \sum_{j=n+1}^{2n} \frac{\partial W_j^k}{\partial x_i} \frac{\partial \Psi_j}{\partial x_i} n_{ij} = \sum_{l=m+1}^{2m} W_l^k B_{im}. \tag{20}$$

In this representation, N was the number of internal points, n was the number of solution centres, m referred to the number of boundary centres in the kth local domain. This process resulted in forming a system of $3N$ sparse linear equations, which could be solved to obtain the stream function values and velocities at internal nodes.

4 STOKES FLOW – NUMERICAL EXAMPLE

The numerical example is a well-known one used to study Stokes flow. The geometrical conditions are formed by a square box filled with a thick liquid. Three sides of this box are still, meaning they do not move (the walls on the sides and the bottom). However, the top side, like a lid, can move. One can see the model configuration in Fig. 2, where we also show how the edges and boundary conditions are set.

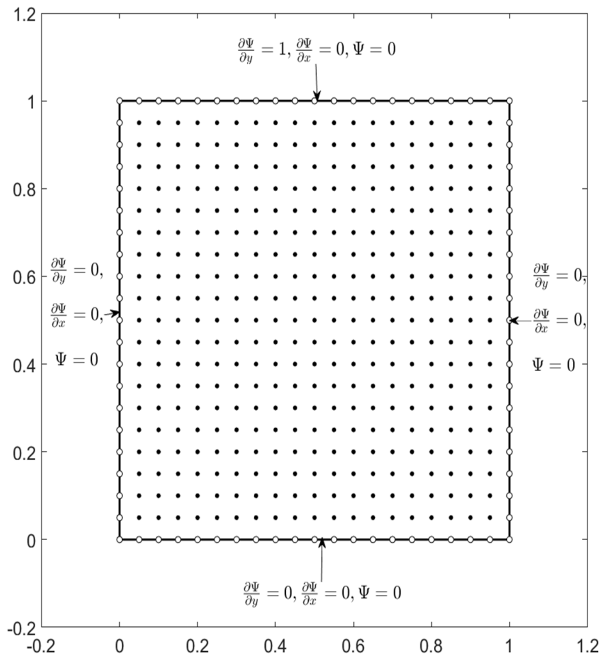

Figure 2: Lid-driven cavity, geometry, and boundary conditions.

In this case, we make the lid move at a steady speed, represented as $u_l = 1$. This motion also makes the liquid inside the box move, especially close to the walls. This creates swirling patterns and flows inside the box because the liquid sticks to the walls.

This numerical experiment is often used as a test problem when they want to try out new computer simulations or experiments [13]. It is a good choice because it is simple, and we know exactly what should happen. This helps to understand how the liquid sticks to the walls, how swirling happens, and when the flow separates from the walls.

In the present study the solution is obtained using three sets of points: one with a grid of 21 × 21 points, another with 41 × 41 points, and the last with 81 × 81 points. Fig. 3 shows the results as lines and arrows representing how the liquid moves at different places in the square cavity.

However, there is no exact solution for this problem in a closed form. So, we compare our results to what other researchers have found using computer codes, like the work in Botella and Peyret [13] and Mužík and Bulko [14]. Table 1 compares the highest and lowest values of scalar stream function ψ" with what they found in Botella and Peyret [13]. Fig. 4 shows the velocity profiles compared with Song et al. [15].

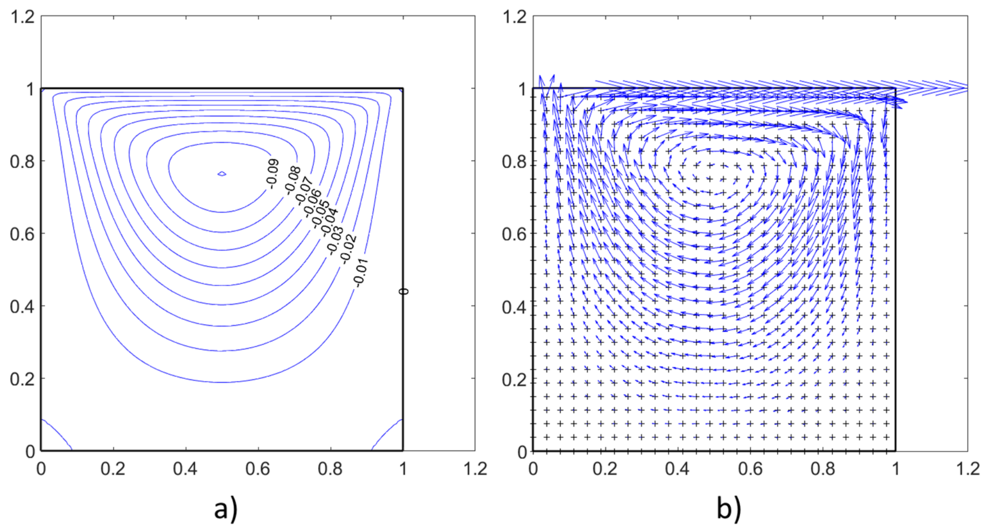

Figure 3: Lid-driven cavity. (a) Contours of ψ; and (b) Velocity vectors.

Table 1: Lid-driven cavity, comparison of ψmin and ψmax.

Mesh	ψ_{min}	ψ_{max}
21 × 21	-9.9788×10^{-2}	0
41 × 41	-9.9961×10^{-2}	1.7403×10^{-6}
81 × 81	-1.0007×10^{-1}	2.2261×10^{-6}
Botella and Peyret [13]	-1.0007×10^{-1}	2.2276×10^{-6}

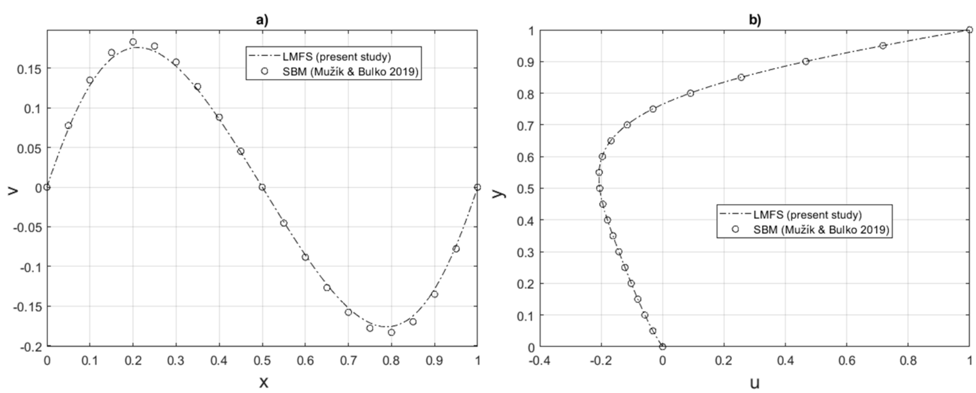

Figure 4: (a) Velocity profile (u) on the centre line at $x = 0.5$; and (b) Velocity profile (v) on the centre line at $y = 0.5$.

5 CONCLUSIONS

This paper introduces a localized method of fundamental solutions for solving the problems related to Stokes flow. The advantage of this localized method of fundamental solutions is that it effectively manages the condition number of the global characteristic matrix, ensuring it remains reasonably well-conditioned. This improvement eliminates one of the primary shortcomings associated with this method. As a result, the overall matrix becomes sparse, maintains good conditioning, and can be solved using standard software tools.

As we continue to advance this method, a natural progression will be to expand its application to three-dimensional tasks or problems that involve non-linear behaviour.

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