Solution of two-dimensional flow in unsaturated media with electrolytic tank

G. Ciaravino & L. Ciaravino
Department of Hydraulic, Geotechnical and Environmental Engineering, University of Naples Federico II, Italy

Abstract

Mathematical models and numerical methods have acquired increasing importance in almost all fields of research. A definitive validation so that these instruments can be fully employed in applications requires that they be subjected to experimental verification. In the present paper the electrolytic tank (based on dissimilar similitude) is used to solve problems of transport phenomena in porous media. In particular, we report a successive approximations method which makes it possible to completely solve any problem of permanent two-dimensional flow in unsaturated aquifers with defined boundary conditions. The experimental technique adopted enables the applicability of the method, already used for one-dimensional flow, to be extended to two-dimensional flow. Moreover, for model analogy the decision to adopt the piezometric head instead of the suction head affords a more substantial overview of the problem defined by the flow equations.
Keywords: numerical method, dissimilar similitude, electrical analogy, electrolytic tank, unsaturated media.

1 Introduction

When defining mathematical models and numerical methods that can interpret complex physical phenomena it is often necessary to make use of simplifications, the consequences of which need to be verified. Simplifications are almost always used to obtain computation procedures that make it possible to achieve sufficiently precise results for technical applications and, as a result, verifying the validity of the mathematical model or numerical method is often the most delicate phase of the study. The validation of a mathematical model and

WIT Transactions on Modelling and Simulation, Vol 53,
www.witpress.com, ISSN 1743-3533 (on-line)
doi:10.2495/BE120201

numerical calculation methods is often performed through comparisons with other mathematical models having the same objective and/or regarding phenomena in which there is an affinity. Such techniques are favoured by the continual evolution of computers available at increasingly competitive prices. It should be noted that in more complex phenomena, where it is critical to define the equation limit conditions, problems regarding the reliability of the final results may be encountered. The most robust validation is therefore achieved either through a test using experimental data from the phenomenon being studied [1–4] (which however are not always readily available) or through experimental data obtained from a physical model, which may incur significant costs. In some cases, it is possible to carry out an experimental verification phase using physical models in dissimilar similitude which yield certainly appreciable results at contained costs [5, 6].

In line with these considerations, the present paper reports a methodology using an electrolytic tank for the solution of complex transport phenomena in porous media. In porous media it is not possible to identify a clearly distinguished surface between the saturated and the fluid-free areas because of capillarity, which results in an intermediate unsaturated area that is difficult to define. In this area the fluid is normally at a lower pressure than that of the surrounding environment (generally air at atmospheric pressure). Moreover, the flow fields of the saturated and the unsaturated areas are dependent on laws that do not entirely coincide.

The electrolytic tank, which is a physical model in dissimilar similitude based on electrical analogies, can be profitably employed to study filtration problems in saturated aquifers. This is achieved by exploiting the coincidence of equations describing the potential flow in a fluid and the passage of electric current through a conductor. In a 2-D potential flow, for instance, defining the velocity potential as Φ and the electric potential as Φ_E (in a Laplacian field), the velocity components are:

$$u = \frac{\partial \Phi}{\partial x} \quad \text{and} \quad v = \frac{\partial \Phi}{\partial y} \;;$$

while the electric current components are:

$$i_x = \frac{1}{\sigma} \cdot \frac{\partial \Phi_E}{\partial x} \quad \text{and} \quad i_y = \frac{1}{\sigma} \cdot \frac{\partial \Phi_E}{\partial y}$$

where σ is the resistivity. The continuity equations are thus, for the fluid:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

and for the electrical system:

$$\frac{\partial i_x}{\partial x} + \frac{\partial i_y}{\partial y} = 0 \;. \tag{2}$$

A comparison of these equations shows that, excluding for now the constant parameter σ, it is possible to identify velocity potential and electric potential, which are components of the velocity and components of the electric current.

Although more complex, it is possible to perform the study on unsaturated aquifers in which the distribution of the piezometric heads is not Laplacian [9, 10]. In the latter case, the aquifer piezometric head distribution is normally only verified, having already been determined by other means.

The present paper reports a successive approximations method, which arguably deserves greater consideration and diffusion in the international scientific community, which makes it possible to achieve the complete solution of any permanent two-dimensional flow problem in unsaturated aquifers whose boundary conditions have been defined.

2 Mathematical definition of the problem

As is well known, flow processes in porous media are regulated by the continuity equation and by Darcy's Law. The general continuity equation in varying flow is:

$$\frac{\partial \varphi}{\partial t} = -div\underline{V} - q \tag{3}$$

where φ is the filling degree, $\underline{V}$ is the filtering velocity and q is a parameter that takes into account any absorption of groundwater by root systems. Furthermore, Darcy's flow equation is:

$$\underline{V} = -k \cdot gradh \tag{4}$$

where k is the permeability coefficient and h is the piezometric head. In dealing with flow in unsaturated media, the suction head τ is usually defined as the opposite of the water expressed in the water column, i.e.: $\tau = -p/\gamma$. In this position, with the z axis upwards, the piezometric head is: $h = z - \tau$. Comparison of equations (3) and (4) provides:

$$\frac{\partial \varphi}{\partial t} = div(k \cdot gradh) - q \tag{5}$$

which in the case of permanent flow and in the absence of root systems reduces to:

$$div(k \cdot gradh) = 0 \ . \tag{6}$$

The differential equation (5) is parabolic and can be integrated analytically only in some cases of one-dimensional flow with particularly simple boundary conditions. Equation (6) can also be analytically integrated in particularly simple cases of one-dimensional flow but, as it is elliptical, it can be interpreted through electrical analogy models such as electrolytic tanks.

www.witpress.com, ISSN 1743-3533 (on-line)

It should be noted that the problem can be solved numerically through mathematical models and suitably programmed computers. The numerical integration of differential equations, however, normally entails problems of accuracy due to truncation, convergence and stability errors that depend on the solution methods and schemes adopted (explicit and implicit methods, finite differences and finite elements procedures) [11–15].

Moreover, unlike in saturated media, in unsaturated media the permeability coefficient k reported in the equations should be considered variable and dependant on the suction head τ; i.e.: $k = k(\tau)$. It should also be remembered that the function $k(\tau)$ presents a cycle of hysteresis due to the way in which the soil is filled and emptied by the fluid. It must therefore be assumed that the flow takes place during the filling or the emptying phase in order to consider k as a univocal function of τ. Under this hypothesis, as $k = k(\tau)$, equation (6) yields:

$$\nabla^2 h = \frac{1}{k}\cdot\frac{dk}{d\tau}\cdot\left[\left(\frac{\partial h}{\partial x}\right)^2+\left(\frac{\partial h}{\partial y}\right)^2+\left(\frac{\partial h}{\partial z}\right)^2-\frac{\partial h}{\partial z}\right]; \tag{7}$$

for the function $k(\tau)$ the expression suggested by Gardner can be assumed (see [16]):

$$k(\tau) = k_S/\left(1+c\cdot\tau^2\right). \tag{8}$$

In this formula the parameters K_S and c, which specifically define the nature of the soil, can in some of the elaborations performed by way of example be set to: K_S = 1cm/day and c = 0.0025 cm^{-2}, as suggested again by Gardner (see [16]). In conclusion, equation (7) can be transformed into:

$$\nabla^2 h = \frac{2c\cdot(z-h)}{1+c\cdot(z-h)^2}\cdot\left[\frac{\partial h}{\partial z}-\left(\frac{\partial h}{\partial x}\right)^2-\left(\frac{\partial h}{\partial y}\right)^2-\left(\frac{\partial h}{\partial z}\right)^2\right] \tag{9}$$

which for two-dimensional flow in the plane (x, z) becomes:

$$\nabla^2 h = \frac{2c\cdot(z-h)}{1+c\cdot(z-h)^2}\cdot\left[\frac{\partial h}{\partial z}-\left(\frac{\partial h}{\partial x}\right)^2-\left(\frac{\partial h}{\partial z}\right)^2\right]. \tag{10}$$

3 Electrolytic tank in Poissonian fields

If the bottom of a two-dimensional electrolytic tank developed in the plane (x, z) is fitted with electrodes supplied by an AC potential $\underline{U}_b(x, z)$ variable from point to point, the electrolyte will be supplied through a capacitive effect with a current of intensity $\underline{i}(x, z)$ equal to:

$$\underline{i}(x,z) = \frac{\theta\cdot\omega\cdot\varepsilon}{\delta}\cdot\underline{U}_b(x,z). \tag{11}$$

where ω is the pulse of the AC potential $\underline{U}_b\,(x,\ z)$, ε is the dielectric constant of the tank bottom of thickness δ and θ is an operator indicating a $\pi/2$ dephasing in advance. This introduction of current corresponds - in the approximation whereby the electrical field in the tank can be considered two-dimensional - to the existence of a distributed current source whose density is given by:

$$\underline{q}(x,z)=\frac{\theta\cdot\omega\cdot\varepsilon}{\delta\cdot s}\cdot\underline{U}_b(x,z)\ . \tag{12}$$

where s is the depth of the electrolyte in the tank. It is well known that for a two-dimensional electrical field with sources distributed in accordance with equation (12), Poisson's equation holds:

$$\nabla^2\underline{U}(x,z)=-\rho\cdot\underline{q}(x,z)=-\frac{\rho\cdot\theta\cdot\omega\cdot\varepsilon}{\delta\cdot s}\cdot\underline{U}_b(x,z)\ . \tag{13}$$

where $\underline{U}\,(x,\ z)$ is the potential distribution in the tank and ρ is the electrical resistivity of the electrolyte.

A comparison of equations (10) and (13) highlights the possibility of making an analogy between filtration flow in an unsaturated medium and the electrical field in the electrolytic tank supplied by capacitors on the tank bottom. In particular, the piezometric head h needs to be placed in relation to the potential $\underline{U}$ by means of an analogy equation:

$$\underline{U}=-\underline{\alpha}\cdot h\ . \tag{14}$$

where $\underline{\alpha}$ is an operative analogy constant. Equations (13) and (14) combined yield:

$$\nabla^2 h=\frac{\rho\cdot\theta\cdot\omega\cdot\varepsilon}{\underline{\alpha}\cdot\delta\cdot s}\cdot\underline{U}_b(x,z)\ . \tag{15}$$

which, combined with equation (10), yields:

$$\frac{\rho\cdot\theta\cdot\omega\cdot\varepsilon}{\underline{\alpha}\cdot\delta\cdot s}\cdot\underline{U}_b(x,z)=\frac{2c\cdot(z-h)}{1+c\cdot(z-h)^2}\cdot\left[\frac{\partial h}{\partial z}-\left(\frac{\partial h}{\partial x}\right)^2-\left(\frac{\partial h}{\partial z}\right)^2\right] \tag{16}$$

from which it is possible to obtain the potential distribution on the bottom of the electrolytic tank $\underline{U}_b\,(x,\ z)$ in the sought analogy.

Equation (16) shows that the voltage to be applied to the bottom of the tank turns out to be a fairly complex function of the pressure head and its spatial derivatives. Two problems in particular arise: the first is that the potential on the bottom of the tank may vary from point to point, making it necessary to have a large number of electrodes supplied at different potentials; the second is the fact that determining the potentials on the bottom would imply prior knowledge of the final solution of the problem. This problem is solved by proposing a successive approximations method.

4 Successive approximations method

As the distribution of the piezometric heads h(x, z) is not known a priori and, therefore, it is not possible to proceed directly through equation (16), a successive approximations method has been set up. In order to reach the objective, equation (16) has been transformed into terms of just electric potentials by means of equation (14) to give the potential distribution on the bottom of the electrolytic tank $\underline{U}_b$ (x, z):

$$\underline{U}_b(x,z)=\frac{\underline{\alpha}\cdot\delta\cdot s}{\rho\cdot\theta\cdot\omega\cdot\varepsilon}\cdot\frac{2c\cdot\left(z+\underline{U}/\underline{\alpha}\right)}{1+c\cdot\left(z+\underline{U}/\underline{\alpha}\right)^2}\cdot\left[\left(\frac{\partial\underline{U}/\underline{\alpha}}{\partial x}\right)^2+\left(\frac{\partial\underline{U}/\underline{\alpha}}{\partial z}\right)^2-\frac{\partial\underline{U}/\underline{\alpha}}{\partial z}\right] \quad (17)$$

This procedure entails reaching the exact potential distribution on the bottom by means of successive adjustments to the electrodes. That is to say, the bottom is subdivided into a sufficiently small number of elements and the potential is adjusted one electrode at a time through a procedure that gradually reaches the actual potential distribution.

First of all a value V_0 is assigned to the ascending velocity of the water on the aquifer surface (at z = 0) which is assumed to be horizontal. On the basis of equation (4) it is possible to set:

$$V_0=-k_S\cdot\left(\frac{\partial h}{\partial z}\right)_0 \quad (18)$$

which from an electrical standpoint on the basis of the analogy equation (14) gives:

$$\frac{\partial\underline{U}}{\partial z}=\underline{\alpha}\cdot\frac{V_0}{k_S}. \quad (19)$$

These equations supply the boundary conditions for the solution of the problem. Let us suppose that, after n successive approximations, we have reached a generic distribution of potentials on the bottom, which we will call $\underline{U}bn$ (x, z). In order to achieve the boundary conditions, the electrode in the tank relative to z = 0 must be at voltage 0 and the derivative with respect to z of the voltage trend, again at z = 0, must have the value of equation (19). In achieving the latter condition, the value of the potential to attribute to the electrode at a pre-fixed ordinate has to be determined (and is obviously calculated by the last approximation performed); let $\underline{U}tn$ be this potential value. Let us call the distribution of potentials in the tank resulting from the supplied power $\underline{U}n$ (x, z). If the distributions $\underline{U}bn$ (x, z) and $\underline{U}n$ (x, z) satisfy equation (17), the problem is solved. If this is not the case, it is necessary to consider one of the elements in which the bottom has been subdivided, at the barycentre of which equation (17) is not verified, and then readjust its potential. This readjustment must be simultaneous with a readjustment of the potential $\underline{U}tn$ as the variations $\underline{U}bn$ (x, z) affect the distribution of $\underline{U}\,n(x, z)$ in the whole tank, while leaving $\underline{U}n$ $(x, 0)$ = 0 and the values of equation (19) equal, again for z = 0. The readjustment must be such as to ensure that equation (17) is verified at the barycentre of the considered element. The new distribution of potentials obtained $\underline{U}b(n+1)$ (x, z), $\underline{U}t(n+1)$ and of potentials in the tank $\underline{U}$ $(n+1)(x, z)$ represents the successive approximation. It has been experimentally found that, provided the elements into which the bottom has been subdivided are sufficiently small, this procedure converges on the solution of the problem.

5 Experimental technique adopted

In order to simplify the experimental procedures it is possible to use more specific measures in order to determine the potential to apply to the bottom element barycentre considered, so that equation (17) is verified.

First, the distribution of the potentials in the tank $\underline{U}_n$ (x, z) obtained after n approximations can be considered, given the linearity of the electrical field equations, as the sum of the distributions $\underline{U}'_n$ (x, z) and $\underline{U}_{Ln}$ (x, z) obtained by supplying the tank for a first time with a bottom voltage equal to $\underline{U}_{bn}$ (x, z) and with the electrodes in the tank earthed and for a second time without powering the bottom and with the electrodes in the tank set to the voltages 0 and $\underline{U}_{tn}$. At this point, in order to determine the values $\underline{U}_{b(n+1)}$ (x, z) and $\underline{U}_{t(n+1)}$ relative to the successive approximation, it is necessary to perform three separate series of measurements.

In the first series the tank needs to be given the potential $\underline{U}'_n$ (x, z) by supplying the bottom with $\underline{U}_{bn}$ (x, z) and with the electrodes in the tank earthed.

In the second series the tank must be supplied only on the electrodes in the water with a potential 0 at the ordinate $z = 0$ and with another potential of our choice $\underline{U}_{t0}$ at the ordinate $z = L$ while disconnecting supply to the bottom. This supply creates a voltage in the tank which we will call $\underline{U}_{L0}$ (x, z).

Finally, the third series calls for the tank to be supplied only through the bottom electrodes element whose potential should be changed and which will be called the reference electrode R. The voltage supply to the latter electrode in this series of measurements will again be a value we choose $\underline{U''}_{b0}$. The electrodes in the tank and all the other electrodes on the bottom must be earthed. This creates a voltage in the tank which we will call $\underline{U''}_{(n+1)}(x, z)$.

Now, similarly to the nth approximation, after $(n+1)$ approximations the potential in the tank $\underline{U}_{(n+1)}(x, z)$ can also be considered as the sum of the voltages $\underline{U'}_{(n+1)}(x, z)$ and $\underline{U}_{L\,(n+1)}(x, z)$ obtained by supplying the tank a first time with the bottom voltages $\underline{U}_{b\,(n+1)}(x, z)$ and with the electrodes in the tank earthed, and a second time without powering the bottom and with the electrodes in the tank set to the voltages 0 and $\underline{U}_{t(n+1)}$.

In turn the potential $\underline{U'}_{(n+1)}(x, z)$ can be considered as the sum of the potential $\underline{U}'_n$ (x, z) and the potential obtained by setting the reference electrode R to a voltage of:

$$\underline{U}_{b\Delta(n+1)} = \underline{U}_{b(n+1)}(x_R, z_R) - \underline{U}_{bn}(x_R, z_R)\ ; \tag{20}$$

if we then make:

$$\underline{U}_{t(n+1)} = \mu_{(n+1)} \cdot \underline{U}_{t0}\ . \tag{21}$$

and

$$\underline{U}_{b\Delta(n+1)} = \lambda_{(n+1)} \cdot \underline{U}''_{b0} ; \tag{22}$$

we can express the voltage in the tank $\underline{U}_{(n+1)}(x, z)$ as:

$$\underline{U}_{(n+1)}(x,z) = \underline{U}'_n(x,z) + \lambda_{(n+1)} \cdot \underline{U}''_{(n+1)}(x,z) + \mu_{(n+1)} \cdot \underline{U}_{L0}(x,z). \tag{23}$$

In equation (23) the two constants $\lambda_{(n+1)}$ and $\mu_{(n+1)}$ are unknown. Moreover, the potential distribution $\underline{U}_{(n+1)}$ (x, z) must satisfy equation (17) at the point of coordinates x_R and z_R, the reference element barycentre R, and equation (19) at all points of the ordinate $x = 0$, that is to say at the aquifer's free surface.

Equations (17) and (19) can therefore be applied in the above-considered points. In these two equations it is possible to measure or fix the value of all the quantities in play except for the two constants $\lambda_{(n+1)}$ and $\mu_{(n+1)}$ which thus represent the only two unknowns and can therefore be easily determined. Once these constants have been determined, the values $\underline{U}_{t(n+1)}$ and $\underline{U}_{b\Delta\ (n+1)}$ and, hence, the voltage distribution $\underline{U}_{b(n+1)}(x, z)$ and $\underline{U}_{(n+1)}(x, z)$ are also determined.

By repeatedly applying the procedure to different bottom elements we get successive voltage distributions that verify equation (17) with increasingly greater accuracy. The successive approximations method can be stopped when equation (17) is held to have been verified all over the bottom with sufficient approximation.

In order to ensure a systematic performance of all three measurements and the calibration of the experimental installation, the circuit reported in figure 1 was set up.

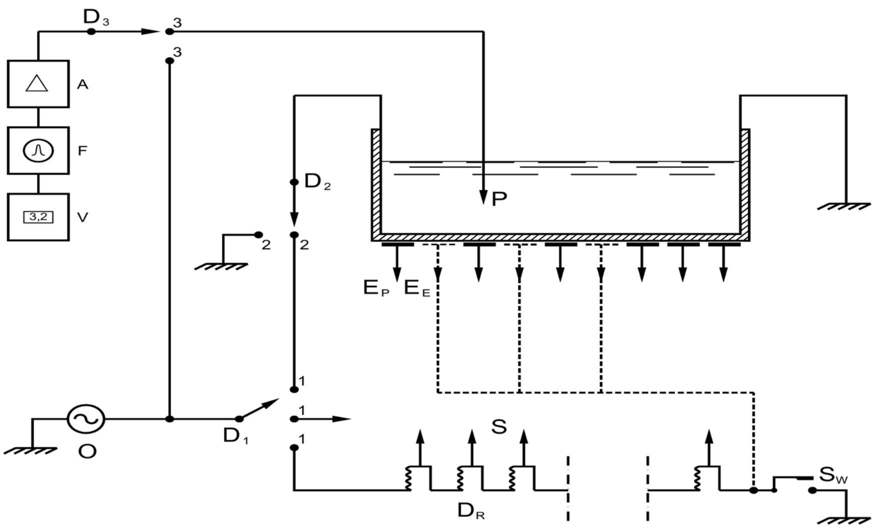

Figure 1: Experimental installation (schema).

The electrodes on the tank bottom can be subdivided into those that should be powered E_P and those that should be earthed E_E during the various measurement or calibration phases. The different potentials to which the electrodes must be connected are obtained by means of the current divider D_R fitted with the sockets S. The resistances of the divider are sufficiently low in order to provide the divider with a much lower output impedance than the tank's input impedance. By means of the deviator switches D_1 and D_2 and the switch S_W, it is possible to create the various power supply combinations by the oscillator O or to pass directly from calibration to the various measurement series. Measurements in the tank make use of the probe P whose signal is first amplified, then filtered onto the power supply frequency in order to eliminate much of the background noise and finally measured with a digital voltmeter. The amplifier's input impedance is much higher than the output impedance from the tank to the probe. The deviator switch D_3 is used to orient the voltmeter to read the tank signal or to check the stability of the supply voltage.

Great care was taken to ensure that the electrodes were properly earthed, when necessary, in order to avoid parasitic returns of capacitive currents. The same care was taken with the shielding and the earthing of the main parts of the electric circuit, as the frequencies adopted (1000 Hz) made it possible for the above mentioned parasitic capacitive voltages to arise.

6 Conclusion

The presented experimental technique entailed successive approximations and made it possible to use a physical model based on dissimilar similitude for the solution of a two-dimensional flow field of waters filtering through unsaturated media. The experimental installation, comprising an electrolytic tank, is based on the electrical analogy which employs the coincidence of equations describing potential flow in a fluid and the passage of electric current in a conductor.

In order to define the model's analogy, we adopted the piezometric head h and not the suction head τ as this enables a more substantial overview of the problem, as is expected in the usual flow equations.

The experimental technique consisted of modifying the distribution of electric potential on the tank bottom, which was subdivided into a number of sufficiently small elements, until the exact value was reached.

Normally this installation only makes it possible to verify the exactness of solutions already obtained by other means.

The described successive approximations procedure, on the other hand, makes it possible to reach a solution of flows that have not been determined in advance, starting from boundary conditions that are not easily defined, such as those relative to flow in unsaturated media (hence in a non-Laplacian field). In this sense, our work shows how the electrolytic tank constitutes a robust aid for the validation of mathematical models and numerical methods for which, in complex situations, an experimental verification is advisable.

It should be noted that the experimental technique adopted can be held to be congruent with the technique adopted numerically in relaxation procedures and with the setting of finite elements mathematical models (in particular in the way of subdividing the bottom into finite dimension elements).

Our experiment shows that the additional error, caused by the size of the number of elements into which the tank bottom is subdivided is somewhat contained. Indeed, we observed an error of less than 4% for a subdivision of the bottom which fell from 100 elements to just 5 elements. It can thus be concluded that the division of the bottom into elements of finite dimensions does not create substantial difficulties in reaching a correct solution of the problem, and thus confirms the validity of experimental technique adopted.

Finally, it should not be forgotten that the interesting results obtainable with these analogical models are also appreciable from an economic point of view: indeed the cost of the installations needed for the experimental activities described appears relatively contained.

References

[1] Green, D.W., Dabiri, H., Weinaug, C.F. and Prill, R., Numerical modelling of unsaturated groundwater flow and comparison of the model to field experiment, *Water Resources Research*, **6 (3)**, 1970.

[2] Vachaud, G., Vauclin, M. and Khanji, D., Etude expérimentale des transferts bidimensionnels dans la zone non saturée. Application à l'étude du drainage d'une nappe à surface libre, *La Houille Blanche*, **1**, pp. 65-74, 1973.

[3] Topp, G.C., Soil water hysteresis in silt loam and clay-loam soils, *Water Resources Research*, **7 (4)**, pp. 914-920, 1971.

[4] Sormail, L. and Vachaud, G., Mèsure des potentiels d'humidité dans les sols non saturés au moyen de tensiométres classiques et osmotiques, *La Houille Blanche*, **8**, pp. 849-860, 1969.

[5] Kline, S.J., *Similitude and Approximation Theory*, New York: Springer-Verlag, 1986.

[6] Liebmann, G., Electrical analogues, *Brit. Journal Appl. Phys.*, **4 (7)**, pp. 193-200, 1953.

[7] Vachaud, G. and Guelin, P., Equations et modéles mathématiques puor le calcul des transferts d'eau dans la zone de sol non saturée, *La Houille Blanche*, **8**, pp. 817-827, 1969.

[8] Thirriot, C., Considérations générales sue le écoulements en milieux non saturés, *La Houille Blanche*, **8**, pp. 805-816, 1969.

[9] Martin, R.J., Masnari, N.A. and Rowe, J.E., Analog representation of Poisson's equation in two dimension, *IRE Trans. Electr. Comp.,* pp. 490, 1966.

[10] Shlomo, N.P., Saturated unsaturated seepage by finite elements, *Proc. ACSE HY*, **12**, 1973.

[11] Schwartzendruber, D., The flow of water in unsaturated soils, In: De Wiest, R. J.M., Flow through porous media, *Academic Press New York and London*, Chp. 6, pp. 215-282, 1969.

[12] Hornberger, G.M. and Remson, I., A moving boundary model of one dimensional saturated unsaturated transient porous flow system, *Water Resources Research*, **6 (3)**, pp. 898-905, 1970.
[13] Freeze, R.A., Influence of the unsaturated flow domain on seepage through earth, *Water Resources Research*, **7 (4)**, pp. 929-941, 1971.
[14] Neuman, S.P., Saturated unsaturated seepage by finite elements, *Proc. ASCE HY*, **12**, pp. 2233-2251, 1973.
[15] Reisenauer, A.E., Methods for solving problems of multidimensional partially saturated steady flow in soils, *Journal of Geophysical Research*, **68 (20)**, pp. 5725-5733, 1963.
[16] De Wiest, R. J.M., Flow through porous media, *Academic Press New York and London*, pp. 238-239, 1969.