A study on topology optimization using the level-set function and BEM

T. Matsumoto, T. Yamada, S. Shichi & T. Takahashi
Nagoya University, Japan

Abstract

A framework to solve topology optimization problems using a level-set based approach and boundary element method (BEM) is presented. The objective functional is assumed to be defined only on the boundary, thus the problem is suitable for handling with BEM. The boundary of the structure is defined as the isosurface of the level-set function. The distribution of the level-set function is obtained by solving an evolution equation defined in a fixed design domain. The evolution equation has a source term related to the topological sensitivity of the objective functional. The sensitivity can be evaluated from BEM analysis results both for the original physical variable and its adjoint variable. The shape of the structure is defined as the isosurface of the level-set function defined in a fixed design domain, and the boundary mesh is generated by extracting the isosurface. Since the fixed design domain has a simple rectangular solid shape and does not change its domain shape through iteration steps, the evolution equation of the level-set function can be solved by FEM more efficiently than by BEM. The effectiveness of the proposed approach is demonstrated through a minimum compliance topology optimization problem.
Keywords: topology optimization, level-set function, adjoint method, boundary element method.

1 Introduction

Because of the current development of fast boundary element algorithms [1, 2], BEM may become a strong analysis tool not only for shape optimization problems but also for topology optimization problems. In particular, BEM is effective for linear problems and those with infinite domains.

WIT Transactions on Modelling and Simulation, Vol 53, © 2012 WIT Press
www.witpress.com, ISSN 1743-3533 (on-line)
doi:10.2495/BE120111

Topology optimization is also a kind of shape optimization problems but allows topology changes in the process of minimizing objective functionals. One solution to handle the topology changes is to model the body with density distributions of materials. This approach has been widely used [3–6, 8], however, finally obtained density distribution corresponding to optimum topology configuration has intermediate values, so-called “gray scales”, between the interior and exterior part of the body.

To circumvent the grayscales, level-set methods have been proposed. Level-set method uses a level-set function that relates points in the design space to different scalar values depending on their locations. The optimum body configuration can be extracted by tracing the values of the level-set function. Level-set methods have also been applied to various engineering applications [9–13]. In particular, the approach by Yamada *et al.* [12] was very effective in controlling the topological complexity by introducing a Tiknohov regularization term [14] for the level-set function. For updating the distribution of the level-set function, a kind of reaction-diffusion equation for the level-set function is used. In these methods, the fixed design domain is analyzed using FEM without extracting the boundary of the actual structure. Generated voids are treated as weak materials of the fixed design domain. Hence, boundary condition cannot be specified on the boundaries resulting from the topology changes.

In this paper, the level-set approach proposed by Yamada *et al.* [12] is used for controlling the configuration of the structure, but BEM is used for analyzing the physical behavior. We assume topology optimization problems appropriate to treat by BEM, hence, the objective functional is defined only on the boundary. The boundary of the structure is defined as the isosurface of the level-set function. The distribution of the level-set function is obtained by solving an evolution equation defined in the fixed design domain. The evolution equation is a kind of reaction-diffusion equation with a source term related to the topological sensitivity of the objective functional. The sensitivity can be evaluated by using the BEM analysis results both for the original physical variable and its adjoint variable. At every iteration step, boundary mesh of the actual structure is generated by extracting the isosurface of the level-set function. Since the fixed design domain has a simple rectangular solid shape and does not change its shape through iteration steps, the evolution equation of the level-set function can be solved by FEM more efficiently rather than by BEM.

The idea of the present formulation is given by considering a potential problem for simplicity, but it can be extended to other problems in the same manner. The effectiveness of the proposed approach is demonstrated through a minimum compliance topology optimization problem.

 © 2012 WIT Press
www.witpress.com, ISSN 1743-3533 (on-line)

2 Formulations

2.1 Boundary element method

To understand the present framework easily, we consider a potential problem governed by Laplace's equation and start from boundary element formulation. Laplace's equation is written as

$$\nabla^2 u = 0 \quad \text{in } \Omega, \tag{1}$$

where u denotes potential and Ω is the domain under consideration.

The boundary conditions are given as follows:

$$u = \bar{u} \quad \text{on } \Gamma_u, \tag{2}$$

$$q = \frac{\partial u}{\partial n} = \bar{q} \quad \text{on } \Gamma_q, \tag{3}$$

where q is the flux, n is the outward normal direction to the boundary, $\bar{u}$ and $\bar{q}$ are the prescribed known functions of u and q on some parts of the boundary Γ_u and Γ_q, respectively.

The integral representation of u at an internal point y corresponding to Eq. (1) is Green's identity [15]:

$$u(y) = \int_\Gamma u^*(x,y)q(x)d\Gamma(x) - \int_\Gamma q^*(x,y)u(x)\,d\Gamma(x), \quad y \in \Gamma, \tag{4}$$

where x denotes points on the boundary while y denotes an arbitrary point in the domain.

$u^*(x,y)$ and $q^*(x,y)$ are the fundamental solution of Laplace's equation and its normal derivative, respectively, given for three-dimensional case by

$$u^*(x,y) = \frac{1}{4\pi r}, \tag{5}$$

$$q^*(x,y) = u^*_{,j}(x,y)n_j(x) = \frac{-1}{4\pi r^2}\frac{\partial r}{\partial n}, \tag{6}$$

where r is the distance between x and y, a subscript denotes a Cartesian component of the corresponding vector, a subscript after a comma denotes a partial differentiation with respect to the coordinate, and summation convention is applied to repeated indices. $n_j(x)$ is the unit outward normal vector at x on the boundary, and $\partial r/\partial n$ denotes the differentiation of r in the direction of the outward normal vector.

Once the boundary potential and flux are obtained, potentials at internal points can be obtained by evaluating the right-hand side of Eq. (4).

The gradients of the potential are also related to a boundary integral representation obtained by differentiating Eq. (4), as follows:

$$u_{,i}(y) = -\int_\Gamma u^*_{,i}(x,y)q(x)d\Gamma(x) + \int_\Gamma u^*_{,ij}(x,y)n_j(x)u(x)\,d\Gamma(x), \quad y \in \Gamma. \tag{7}$$

The unknown potential and flux on the boundary are obtained by solving the following boundary integral equation which is the limit of $y \in \Omega$ to $y \in \Gamma$.

$$Cu(y) + \int_\Gamma q^*(x,y)u(x)\, d\Gamma(x) = \int_\Gamma u^*(x,y)q(x) d\Gamma(x), \quad y \in \Gamma, \tag{8}$$

where C is a constant, resulting in $1/2$ when y lies on a smooth part of the boundary. Discretizing Eq. (8), we have the following system of algebraic equations:

$$[H]\{u\} = [G]\{q\}. \tag{9}$$

Rearranging Eq. (9) so that all the unknowns come to left-hand side and all the others to the right-hand side results in

$$[A]\{X\} = \{Y\}, \tag{10}$$

where $\{X\}$ is the vector consisting only of unknown nodal values, while $\{Y\}$ is the vector obtained by multiplying the known nodal values with corresponding parts of the coefficient matrices $[H]$ and $[G]$.

2.2 Shape control based on level-set function

Level-set function is a scalar function of point and defines the domain and boundary corresponding to its value. We consider a domain D, so-called "fixed design domain," which is fixed through iterative optimization process. The actual physical domain Ω that should finally be obtained as an optimum shape is supposed to exist within D. Let the boundary of Ω be denoted by Γ. Then, in this study, we define a level-set function defined by

$$\begin{cases} 0 < \phi(\mathbf{x}) \leq 1, & \mathbf{x} \in \Omega \setminus \Gamma \\ \phi(\mathbf{x}) = 0, & \mathbf{x} \in \Gamma \\ -1 \leq \phi(\mathbf{x}) < 0, & \mathbf{x} \in D \setminus \Omega \end{cases}. \tag{11}$$

Once a distribution of ϕ is obtained, the domain Ω can be determined inversely by using a characteristic function as

$$\chi_\phi(\mathbf{x}) \equiv \chi(\phi(\mathbf{x})) = \begin{cases} 1 & \text{if } \phi(\mathbf{x}) \geq 0 \\ 0 & \text{if } \phi(\mathbf{x}) < 0 \end{cases}. \tag{12}$$

2.3 Topology optimization

We consider a topology optimization problem with a volume constraint as

$$\inf_\phi \qquad F(\chi_\phi) = \int_\Gamma f(u,q)\, d\Gamma, \tag{13}$$

$$\text{subject to} \quad I = \int_D \mu \left(\nabla^2 u\right) \chi_\phi \, d\Omega = 0, \tag{14}$$

$$G(\chi_\phi) = \int_D \chi_\phi \, d\Omega - G_{\max} \leq 0. \tag{15}$$

where $f(u, q)$ is a functional of u and q defined on Γ or some part of Γ, μ is a Lagrange multiplier, which is treated as an adjoint variable later, and $G_{\max}$ is the admissible upper limit of the volume of the domain.

The Lagrangian of the objective functional F becomes

$$\bar{F} = F + I + \lambda G, \tag{16}$$

where λ denotes a Lagrange multiplier for the volume constraint inequality.

The necessary condition for the optimality of $\bar{F}$ for a topology variation is the following Karush-Kuhn-Tucker (KKT) condition:

$$F' + I' + \lambda = 0, \quad I = 0, \quad \lambda G(\chi_\phi) = 0, \quad \lambda \geq 0, \quad G(\chi_\phi) \leq 0, \tag{17}$$

where a prime symbol ($'$) denotes a topological derivative.

In the present study, instead of using the above KKT condition to obtain the optimum topology, we solve an evolution equation obtained by assuming that the time evaluation of the level-set function is proportional to the topological derivative of the Lagrangian. Since the characteristic function χ allows discontinuity everywhere, we may have a very complicated topology as the optimum solution. In order to regularize the illposedness of the present topology optimization problem, we add a term based on Tikhonov's regularization [12, 14] as

$$\bar{F}_R = \bar{F} + \int_D \tau \left|\nabla^2 \phi\right| \, d\Omega, \tag{18}$$

where τ is a positive regularization parameter. With this regularization term, the curvature distribution of the level-set function can be controlled so that the obtained topology has a certain simplicity.

By introducing a fictitious time, we assume that the variation of the level-set function with respect to the fictitious time is proportional to the topological derivative of $\bar{F}_R$, as

$$\frac{\partial \phi}{\partial t} = -K \bar{F}_R' \tag{19}$$

Using Eq. (18) in Eq. (19), we obtain an evolution equation from the topological derivative of $\bar{F}_R$ for a point in D as follows:

$$\frac{\partial \phi}{\partial t} = -K \left(F' + I' + \lambda - \tau \nabla^2 \phi\right), \tag{20}$$

where K is a constant.

 © 2012 WIT Press
www.witpress.com, ISSN 1743-3533 (on-line)

2.4 Topological derivative of the Lagrangian

In order to evaluate the topological derivative in Eq. (20), we define an adjoint system. Let us consider the first and second terms of Eq. (16) as

$$\begin{aligned} J \equiv F + I &= \int_\Gamma f(u,q)\,d\Gamma + \int_D \mu \nabla^2 u \chi_\phi \, d\Omega \\ &= \int_\Gamma f(u,q)\,d\Gamma + \int_\Omega \mu \nabla^2 u\, d\Omega. \end{aligned} \tag{21}$$

We evaluate the variation of J caused by the variation of ϕ effecting a deletion of an infinitely small spherical domain Ω_ε from Ω via the corresponding variation of χ.

J can be converted into the following form by integrating the second term by parts once:

$$J(\chi) = \int_\Gamma f(u,q)\,d\Gamma + \int_\Gamma \mu q\, d\Gamma - \int_\Omega \nabla\mu \cdot \nabla u\, d\Omega. \tag{22}$$

Let us assume that $f(u,q)$ is not defined on the boundary Γ_ε of Ω_ε. Then the objective functional J for this topology change can be written as

$$\begin{aligned} J(\chi - \delta\chi) = &\int_\Gamma \left(f(u,q) + \frac{\partial f}{\partial u}\delta u + \frac{\partial f}{\partial q}\delta q \right) d\Gamma + \int_\Gamma \mu\left(q + \delta q\right) d\Gamma \\ &+ \int_{\Gamma_\varepsilon} \mu\left(q+\delta q\right) d\Gamma - \int_{\Omega \setminus \Omega_\varepsilon} \nabla\mu \cdot \big(\nabla u + \nabla(\delta u)\big)\, d\Omega. \end{aligned} \tag{23}$$

Thus the variation of J becomes

$$\begin{aligned} \delta J = &\int_{\Gamma_u \cup \Gamma_q} \left(\mu + \frac{\partial f}{\partial q}\right) \delta q\, d\Gamma - \int_{\Gamma_u \cup \Gamma_q} \left(\frac{\partial \mu}{\partial n} - \frac{\partial f}{\partial u}\right) \delta u\, d\Gamma \\ &+ \int_{\Gamma_\varepsilon} \mu(q+\delta q)\, d\Gamma - \int_{\Gamma_\varepsilon} \frac{\partial \mu}{\partial n}\delta u\, d\Gamma + \int_{\Omega\setminus\Omega_\varepsilon} (\nabla^2 \mu)\delta u\, d\Omega \\ &+ \int_{\Omega_\varepsilon} \nabla\mu\cdot\nabla u\, d\Omega. \end{aligned} \tag{24}$$

Although the variations δu on Γ_u and δ_q on Γ_q are known because u and q are prescribed as the boundary condition on Γ_u and Γ_q, respectively, the variations δu on Γ_q, δ_q on Γ_u, and δ_u in $\Omega \setminus \Omega_\varepsilon$ are unknown. To eliminate these unknown quantities, we use μ satisfying the following boundary value problem for the adjoint variable μ:

$$\nabla^2 \mu = 0 \qquad \text{in } \Omega, \tag{25}$$

$$\mu = -\frac{\partial f}{\partial q} \qquad \text{on } \Gamma_u, \tag{26}$$

$$\frac{\partial \mu}{\partial n} = \frac{\partial f}{\partial u} \quad \text{on } \Gamma_q. \tag{27}$$

Then, Eq. (24) becomes

$$\delta J = \int_{\Gamma_q} \left(\mu + \frac{\partial f}{\partial q} \right) \delta q \, d\Gamma - \int_{\Gamma_u} \left(\frac{\partial \mu}{\partial n} - \frac{\partial f}{\partial u} \right) \delta u \, d\Gamma \\ + \int_{\Gamma_\varepsilon} \mu(q + \delta q) \, d\Gamma - \int_{\Gamma_\varepsilon} \frac{\partial \mu}{\partial n} \delta u \, d\Gamma + \int_{\Omega_\varepsilon} \nabla \mu \cdot \nabla u \, d\Omega. \tag{28}$$

Since $u \in \Gamma_u$ and $q \in \Gamma_q$ are given as boundary conditions, their values do not change under a topology change, and the corresponding boundary integrals vanish. It can easily be shown that the third and fourth terms on the right-hand side of Eq. (28) vanish when ε is sufficiently small under the assumption that u is differentiable at the center of Ω_ε. Also, the last term of Eq. (28) can be evaluated as

$$\int_{\Omega_\varepsilon} \nabla \mu \cdot \nabla u \, d\Omega \approx \left(\int_{\Omega_\varepsilon} d\Omega \right) \nabla \mu^0 \cdot \nabla u^0 = \delta \Omega_\varepsilon \left(\nabla \mu^0 \cdot \nabla u^0 \right), \tag{29}$$

where $\delta\Omega_\varepsilon$ is the volume of Ω_ε, $\nabla \mu^0$ and ∇u^0 are the gradients of μ and u at the center of Ω_ε, respectively.

Thus, the topological derivative of J is obtained as follows:

$$J' = \lim_{\varepsilon \to 0} \frac{\delta J}{\delta \Omega_\varepsilon} = \nabla \mu^0 \cdot \nabla u^0. \tag{30}$$

Finally, the evolution equation (20) becomes

$$\frac{\partial \phi}{\partial t} = -K \left(\nabla \mu \cdot \nabla u + \lambda - \tau \nabla^2 \phi \right), \tag{31}$$

where $\nabla \mu^0$ and ∇u^0 are simply written as $\nabla \mu$ and ∇u, respectively.

3 Computing algorithm

Since the shape of the domain is obtained by the distribution of the level-set function ϕ via χ_ϕ, we have to solve Eq. (31) starting from the initial distribution of ϕ corresponding to the initial configuration of Ω. The domain D is defined so that it contains the whole domain Ω. Therefore, a domain of a rectangular solid can be employed for D. The derivative of ϕ with respect to the virtual time t of Eq. (31) is discretized using a finite difference, then FEM can be used to solve Eq. (31) for the domain D. Because the shape of D is so simple and FEM mesh does not change even when the topology of Ω changes, this FEM calculation is straightforward and easy. However, $\nabla \mu$ and ∇u are the solutions of the boundary value problems for $\Gamma = \partial\Omega$ that has basically a complicated shape. Hence, we use BEM for calculating $\nabla \mu$ and ∇u at internal points of Ω because its boundary Γ

 © 2012 WIT Press
www.witpress.com, ISSN 1743-3533 (on-line)

can be generated easily from the distribution of ϕ by extracting its isosurface and the computation accuracy of the gradients is rather high in BEM. The computation algorithm of the present topology optimization framework is shown below.

(1) Initialize ϕ.
(2) Generate boundary mesh.
(3) BEM analysis for u to calculate ∇u at internal points in Ω.
(4) BEM analysis for μ to calculate $\nabla \mu$ at internal points in Ω.
(5) Evaluate objective functional F. If converged, exit from computation.
(6) Evaluate sensitivities of objective functional F'.
(7) Update level-set function ϕ using FEM.
(8) Go to step (2).

4 Numerical example

We show a topology optimization example based on the above stated procedure for linear elastostatic solid. The governing equation and the boundary condition corresponding to Eqs.(1), (2), and (3) are those corresponding to linear elastostatic problems, i.e., Navier's equation and boundary conditions for displacement and traction, respectively. Somigliana's identity is used instead of Green's identity, and all the other representations can be derived in the same way.

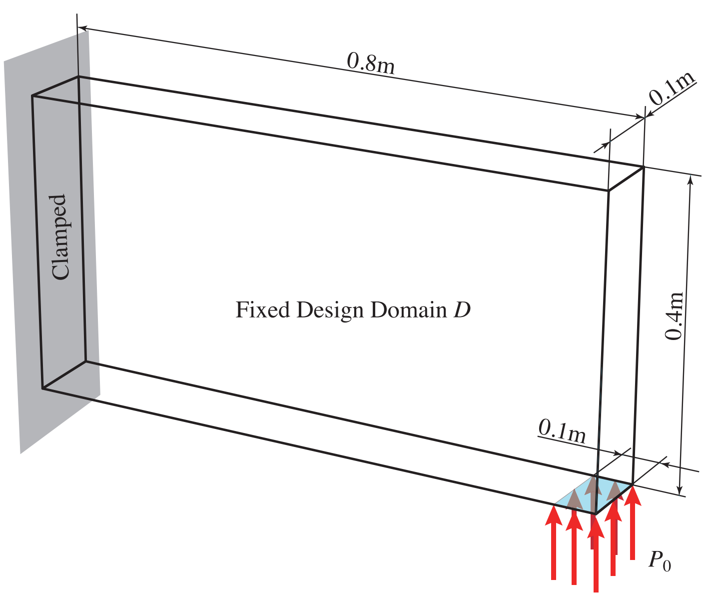

Figure 1: Example fixed design domain and boundary conditions.

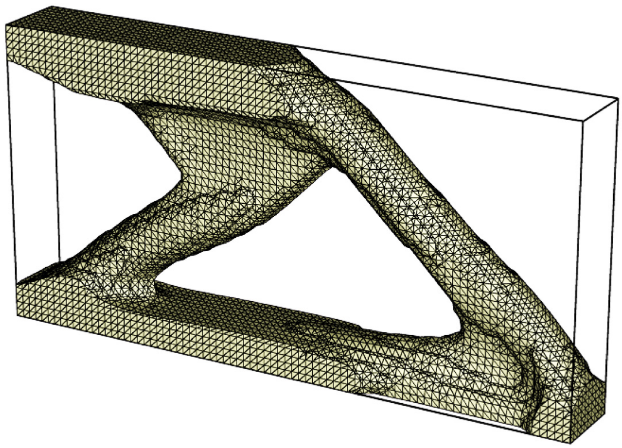

Figure 2: Obtained optimal configuration.

We consider a minimum compliance problem for a rectangular solid plate domain, clamped on one end and subjected to a uniform traction load $\bar{t}_i$ whose magnitude is P_0 on the area of $0.1\mathrm{m} \times 0.4\mathrm{m}$, as shown in Figure 1. Young's modulus and Poisson's ratio are assumed as 216GPa and 0.3, respectively. In this case, the objective functional given as a boundary integral becomes

$$F = \int_{\Gamma_p} \bar{t}_i u_i \, d\Gamma. \tag{32}$$

This domain is considered as the fixed design domain and discretized with $80 \times 40 \times 10$ hexahedral finite elements uniformly. Linear shape functions are used to update the level-set function by FEM. The boundary elements are generated easily by calculating the cross sections of the hexahedral elements satisfying $\phi = 0$ at every updating step. The cross sections corresponding to isosurface for $\phi = 0$ are divided into triangular linear elements to use for subsequent BEM analyses.

The upper limit of the volume constraint $G_{\max}$ is set to 40% of the fixed design domain. The regularization parameter is chosen as $\tau = 0.0 \times 10^{-4}$.

In Figure 2 is shown an optimal configuration of the solid. As can be seen, a shape with a topology different from that of starting shape is obtained. The complexities of the shape and topology change in accordance with the regularization parameter τ's value. We find that the topology optimization using the present framework with BEM may be effective and promising to extend for various engineering applications.

5 Concluding remarks

A framework to solve topology optimization problems using level-set function and BEM has been presented. The shape of the structure is defined as the isosurface of the level-set function defined in a fixed design domain. The distribution of the level-set function is governed by a reaction-diffusion equation with a source term that is related to the topological sensitivity of the objective functional. The boundary mesh is generated from the isosurface of the level-set function, and the BEM analysis is performed for the original physical quantity and its adjoint variable both defined on this isosurface to evaluate the source term of the reaction diffusion equation. The derived adjoint variable approach was applied to a numerical example of minimum compliance topology optimization problem to validate its effectiveness.

References

[1] Rokhlin, V. Rapid solution of integral equations of classical potential theory. *Journal of Computational Physics*, **60**(2), pp. 187-207, 1985.
[2] Yoshida, K. PhD Thesis, Kyoto University, Japan, 2001.
[3] Yang R, Chuang C. Optimal topology design using linear programming. *Computers & Structures*, **52**(2), pp. 265–275, 1994.
[4] Bendsøe M. Optimal shape design as a material distribution problem. *Structural and multidisciplinary optimization*, **1**(4), pp. 193–202, 1989.
[5] Bendsøe M, Sigmund O. *Topology optimization: theory, methods, and applications*. Springer, 2003.
[6] Bendsøe M, Kikuchi N. Generating optimal topologies in structural design using a homogenization method. *Computer methods in applied mechanics and engineering*, **71**(2), pp. 197–224, 1988.
[7] Suzuki K, Kikuchi N. A homogenization method for shape and topology optimization. *Computer Methods in Applied Mechanics and Engineering*, **93**(3), pp. 291–318, 1991.
[8] Allaire G, Kohn R. Optimal design for minimum weight and compliance in plane stress using extremal microstructures. *European journal of mechanics. A. Solids*, **12**(6), pp. 839–878, 1993.
[9] Wang M, Wang X. "color" level sets: a multi-phase method for structural topology optimization with multiple materials. *Computer Methods in Applied Mechanics and Engineering*, **193**(6-8), pp. 469–496, 2004.
[10] Luo J, Luo Z, Chen S, Tong L, Wang M. A new level set method for systematic design of hinge-free compliant mechanisms. *Computer Methods in Applied Mechanics and Engineering*, **198**(2), pp. 318–331, 2008.
[11] Allaire G, Jouve F. Minimum stress optimal design with the level set method. *Engineering analysis with boundary elements*, **32**(11), pp. 909–918, 2008.
[12] Yamada T, Izui K, Nishiwaki S, Takezawa A. A topology optimization method based on the level set method incorporating a fictitious interface

 © 2012 WIT Press
www.witpress.com, ISSN 1743-3533 (on-line)

energy. *Computer Methods in Applied Mechanics and Engineering*, **199**(45-48), pp. 2876–2891, 2010.

[13] Yamada T, Izui K, Nishiwaki S. A level set-based topology optimization method for maximizing thermal diffusivity in problems including design-dependent effects. *Journal of Mechanical Design*, **133**(031011), 2011.

[14] Tikhonov A, Arsenin V, John F. *Solutions of ill-posed problems*. Winston & Sons, Washington, DC, 1977.

[15] Brebbia, CA, Telles, JCF and Wrobel, LC. *Boundary Element Techniques: Theory and Applications in Engineering*, Springer, 1984.