Boundary element modelling of non-linear buckling for symmetrically laminated plates

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Abstract

The non-linear buckling of composite laminates, triggered by geometric imperfections, is here analysed adopting a boundary element methodology. The non-linear theory for thin anisotropic plates couples in-plane forces causing buckling with the consequent bending deformation. The adopted formulation for in-plane forces in terms of the stress function is mathematically identical to that for the bending problem, thus boundary integral equations and fundamental solutions of the same form are used. Differential equations governing increments of the stress function and the deflection are obtained; the resulting integral equations include irreducible domain integrals depending on powers or products of the second derivatives of the stress function and the deflection. The latter are calculated through complementary integral equations obtained from the original ones by differentiating the field variables in the domain. The solution process accounts for the domain integrals through an iterative scheme; thus there is no need for modelling any field variables in the plate domain although domain meshing is necessary for performing numerical integrations. The boundary is meshed into quadratic discontinuous elements while the domain is divided into triangular cells with linear discontinuous variation of the relevant field variables for integration purposes. The analysis is implemented through a suit of C codes and applied to a rectangular symmetrically laminated plate. Its predictions are compared with published results obtained by other methods. The accuracy and limitations of the formulation are discussed and alternative approaches for expanding its scope pointed out.

Keywords: boundary elements, composites, laminates, postbuckling, initial imperfections, irreducible domain integrals.

www.witpress.com, ISSN 1743-355X (on-line)
doi:10.2495/BE090191

1 Introduction

Laminated plate structures are used for a variety of functions in aerospace and other industries due to their favourable strength-to-weight ratio. During the fabrication of such structures, it is possible that small deviations from the intended, perfectly flat shapes may occur. These deviations, known as geometric imperfections, may alter significantly the buckling behaviour of a plate. The plate response to in-plane loads in the presence of imperfections is governed by a non-linear thin-plate theory which can predict the large deflections occurring at or above the critical buckling load.

Early numerical studies on the post-buckling behaviour of perfectly flat, rectangular orthotropic plates were based on series solutions for the deflection and the stress function [1–3]. This approach was extended to include the flexure-extension coupling terms as well as the effects of transverse shear and initial imperfections [4]. Shear deformation effects have also been studied through finite element analyses based on higher-order theories [5]. Such analyses have also allowed the assessment of boundary conditions [6]. Stiffened laminated panels have been analysed by a spline finite strip method also accounting for shear deformation [7].

The boundary element method (BEM) had originally been applied to the analysis of non-linear isotropic plate behaviour induced by high lateral loads [8]. Other incremental and iterative approaches extended the scope of BEM to predict the non-linear plate response to in-plane edge loading. Such a response can either be generated as a bifurcation from the fundamental equilibrium path at the critical load [9, 10] or initiated by imperfections [11]. There has not been any known attempt at analysing the post-buckling behaviour of anisotropic laminated plates using a BEM-based technique.

An analytical procedure is presented here whereby BEM modelling, based on classical plate theory, is combined with irreducible domain integrals involving the deflection curvatures and the membrane stresses. The non-linear plate response to in-plane loading initiated by imperfections is determined incrementally with the non-linear terms taken into account through iterations within each loading step. The proposed boundary element formulation for the post-buckling analysis of symmetrically laminated plates has the advantage of using the same type of fundamental solution and its derivatives for both the membrane stress and buckling analyses. This allows the use of non-uniform edge loads for the analysis while facilitating the programming effort.

2 Non-linear plate theory

The orientation of a plate is assumed such that its mid-plane coincides with the x_1-x_2 plane of a two-dimensional Cartesian frame of reference. Greek indices for the range from 1 to 2 and the summation convention for terms with repeated indices are adopted. Plates with initial imperfections $w^i(x_\alpha)$, when loaded in their plane, undergo some deflection $w(x_\alpha)$ before the theoretical critical buckling load is reached. The total plate deflection is given by

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$$\hat{w} = w + w^i$$

The developing in-plane forces $N_{\alpha\beta}$ and bending moments $M_{\alpha\beta}$ satisfy the equations of equilibrium

$$N_{\alpha\beta,\beta} + f_\alpha = 0 \tag{1}$$

$$M_{\alpha\beta,\alpha\beta} + (N_{\alpha\beta}\, \hat{w}_{,\alpha})_{,\beta} + q = 0 \tag{2}$$

where $q(x_\alpha)$ is a lateral distributed force acting on the middle plane and f_α the body force, which may be derivable from a potential function V according to

$$f_\alpha = - V_{,\alpha} \tag{3}$$

The solution of eqns (1) and (2) should satisfy the boundary conditions

$$n_\beta N_{\alpha\beta} = \tilde{p}_\alpha \ \text{ or } u_\alpha = \tilde{u}_\alpha \tag{4}$$

$$n_\alpha M_{\alpha\beta,\beta} + s_\alpha s_\beta n_\kappa M_{\alpha\kappa,\beta} + \phi s_\alpha s_\beta M_{\alpha\beta} + n_\beta N_{\alpha\beta}\, \hat{w}_{,\alpha} = \tilde{V}_n + \tilde{p}_\alpha \hat{w}_{,\alpha} \ \text{ or } w = \tilde{w} \tag{5}$$

$$n_\alpha n_\beta M_{\alpha\beta} = \tilde{M}_n \ \text{ or } n_\alpha w_{,\alpha} = \tilde{\theta}_n \tag{6}$$

$$[\![s_\alpha n_\beta M_{\alpha\beta}]\!]_j = \tilde{C}_j \ \text{ or } w_j = \tilde{w}_j \tag{7}$$

where u_α are the in-plane displacements, n_α, s_α are, respectively, the direction cosines of unit vectors normal and tangent to the plate boundary, ϕ is the boundary curvature; $\tilde{u}_\alpha$, $\tilde{w}$, $\tilde{\theta}_n$ are, respectively, prescribed boundary values of the in-plane displacements, deflection and deflection gradient relative to the boundary normal; $\tilde{p}_\alpha$, $\tilde{V}_n$, $\tilde{M}_n$, $\tilde{C}_j$ are, respectively, prescribed boundary values of the traction, shear force, bending moment in the plane of the boundary normal, force at boundary corner j.

The constitutive equations for symmetrically laminated plates are

$$N_{\alpha\beta} = A_{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta} \tag{8}$$

$$M_{\alpha\beta} = D_{\alpha\beta\gamma\delta}\kappa_{\gamma\delta} \tag{9}$$

where the mid-plane strains $\varepsilon_{\alpha\beta}$ and deflection curvatures $\kappa_{\alpha\beta}$ are related to the displacements by

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha} + \hat{w}_{,\alpha}\ \hat{w}_{,\beta} - w^i{}_{,\alpha}\ w^i{}_{,\beta}) \tag{10}$$

$$\kappa_{\alpha\beta} = -w_{,\alpha\beta} \tag{11}$$

The compatibility condition is obtained by eliminating the in-plane displacements from eqn (10). This is accomplished by defining the operator

$$L_{\alpha\beta} = \delta_{\alpha\beta}\nabla^2 - \partial_\alpha \partial_\beta \tag{12}$$

and applying it to both sides of eqn (10) leading to:

$$L_{\alpha\beta}\varepsilon_{\alpha\beta} = \frac{1}{2}[\, L_{\alpha\beta}(\,\hat{w},_{\alpha}\ \hat{w},_{\beta}\,) - L_{\alpha\beta}(\,w^{i},_{\alpha}\ w^{i},_{\beta}\,)] \tag{13}$$

It can be shown that

$$L_{\alpha\beta}(w,_{\alpha}\, w,_{\beta}) \ = -\,(L_{\alpha\beta}\, w)w,_{\alpha\beta} \tag{14}$$

Hence the compatibility condition, eqn (13), becomes

$$L_{\alpha\beta}\varepsilon_{\alpha\beta} = -\frac{1}{2}[L_{\alpha\beta}(\,\hat{w}\,)\,\hat{w},_{\alpha\beta} - L_{\alpha\beta}(\,w^{i}\,)\,w^{i},_{\alpha\beta}\,] \tag{15}$$

Defining $A^{-1}_{\alpha\beta\gamma\delta}$ as the inverse of $A_{\alpha\beta\gamma\delta}$:

$$A^{-1}_{\alpha\beta\gamma\delta}\, A_{\gamma\delta\lambda\mu} = \delta_{\alpha\lambda}\delta_{\beta\mu}, \tag{16}$$

constitutive eqn (8) is re-arranged to

$$\varepsilon_{\alpha\beta} \ = A^{-1}_{\alpha\beta\gamma\delta}\, N_{\gamma\delta}. \tag{17}$$

Introducing the stress function F such that

$$N_{\alpha\beta} = \ L_{\alpha\beta}F + \delta_{\alpha\beta}\, V, \tag{18}$$

compatibility eqn (15) can be expressed in terms of F and w:

$$L_{\alpha\beta}[\ A^{-1}_{\alpha\beta\gamma\delta}\ \ (L_{\gamma\delta}F + \delta_{\gamma\delta}\, V) = -\frac{1}{2}[L_{\alpha\beta}(\,\hat{w}\,)\,\hat{w},_{\alpha\beta} - L_{\alpha\beta}(\,w^{i}\,)\,w^{i},_{\alpha\beta}\,] \tag{19}$$

Using constitutive eqn (9), curvature-deflection relation (11) and expressing the in-plane forces in terms of the stress function according to eqn (18), equilibrium eqn (2) can also be written in terms of F and w:

$$-\,D_{\alpha\beta\gamma\delta}\, w,_{\alpha\beta\gamma\delta} + [(L_{\alpha\beta}F + \delta_{\alpha\beta}\, V)\,\hat{w},_{\alpha}\,],_{\beta} + q = 0 \tag{20}$$

In the absence of body forces and lateral pressure, the only external action is that due to in-plane tractions causing non-linear buckling. Thus, field equations (19) and (20) are reduced to

$$\hat{A}_{\alpha\beta\gamma\delta}\, F,_{\alpha\beta\gamma\delta} = -\frac{1}{2}[L_{\alpha\beta}(\,\hat{w}\,)\,\hat{w},_{\alpha\beta} - L_{\alpha\beta}(\,w^{i}\,)\,w^{i},_{\alpha\beta}\,] \tag{21}$$

$$D_{\alpha\beta\gamma\delta}\, w,_{\alpha\beta\gamma\delta} = (L_{\alpha\beta}F)\ \ \hat{w},_{\alpha\beta} \tag{22}$$

where

$$\hat{A}_{\alpha\beta\gamma\delta} \ = \ A^{-1}_{\kappa\kappa\lambda\lambda}\ \delta_{\alpha\beta}\delta_{\gamma\delta} - \ A^{-1}_{\alpha\beta\kappa\kappa}\ \delta_{\gamma\delta} - \ A^{-1}_{\kappa\kappa\gamma\delta}\ \delta_{\alpha\beta} + A^{-1}_{\alpha\beta\gamma\delta} \tag{23}$$

An incremental procedure is adopted for the solution of the non-linear system of eqns (21) and (22). The two field variables are incremented by small amounts such that $F \to F + \delta F$, $w \to w + \delta w$ and the field equations governing the increments are derived as follows

$$\hat{A}_{\alpha\beta\gamma\delta}\,\delta F,_{\alpha\beta\gamma\delta} = -L_{\alpha\beta}(\hat{w})\delta w,_{\alpha\beta} + \frac{1}{2}L_{\alpha\beta}(\delta w)\delta w,_{\alpha\beta} \tag{24}$$

$$D_{\alpha\beta\gamma\delta}\,\delta w,_{\alpha\beta\gamma\delta} = (L_{\alpha\beta}\,\hat{w})\delta F,_{\alpha\beta} + (L_{\alpha\beta}F)\delta w,_{\alpha\beta} + (L_{\alpha\beta}\delta F)\delta w,_{\alpha\beta} \tag{25}$$

The field variable increments should also satisfy the boundary conditions

$$\delta p_\alpha = n_\beta \delta N_{\alpha\beta} = \delta\tilde{p}_\alpha \ \text{ or } \ \delta u_\alpha = \delta\tilde{u}_\alpha \tag{26}$$

$$\delta V_n + \delta p_\alpha\,\hat{w},_\alpha + (p_\alpha + \delta p_\alpha)\delta w,_\alpha = \delta\tilde{V}_n \ \text{ or } \ \delta w = \delta\tilde{w} \tag{27}$$

$$\delta M_n = \delta\tilde{M}_n \ \text{ or } \ \delta\theta_n = \delta\tilde{\theta}_n \tag{28}$$

$$\delta C_j = [\![\delta M_{ns}]\!]_j = \delta\tilde{C}_j \ \text{ or } \ \delta w_j = \delta\tilde{w}_j \tag{29}$$

which are derived from eqns (4)-(7).

3 Boundary integral equations for the linear operators

The left-hand sides of both field eqns (21) and (22) or both incremental eqns (24) and (25) involve linear operators in the form

$$\Lambda^c(u) = C_{\alpha\beta\gamma\delta}\,u,_{\alpha\beta\gamma\delta} \tag{30}$$

where $C_{\alpha\beta\gamma\delta}$ is a symmetric fourth-order constant tensor and $u(x_1,x_2)$ a function defined in the two-dimensional domain Ω bounded by contour Γ. Due to the symmetry of $C_{\alpha\beta\gamma\delta}$, the reciprocity relation

$$\int_\Omega C_{\alpha\beta\gamma\delta} u,_{\alpha\beta}\, u^*,_{\gamma\delta}\, \mathrm{d}\Omega = \int_\Omega C_{\alpha\beta\gamma\delta} u^*,_{\alpha\beta}\, u,_{\gamma\delta}\, \mathrm{d}\Omega \tag{31}$$

can be derived, where $u^*(x_1,x_2)$ is a second function, also defined in Ω. Integrating by parts both sides of eqn (31), applying Green's theorem and defining the operators

$$M_n^c(u) = -C_{\alpha\beta\gamma\delta} n_\alpha n_\beta\, u,_{\gamma\delta} \tag{32}$$

$$M_{ns}^c(u) = -C_{\alpha\beta\gamma\delta} n_\alpha s_\beta\, u,_{\gamma\delta} \tag{33}$$

$$V^c(u) = -(C_{\alpha\beta\gamma\delta} n_\alpha + C_{\alpha\kappa\gamma\delta} s_\alpha s_\beta n_\kappa)\, u,_{\beta\gamma\delta} - \phi\, C_{\alpha\beta\gamma\delta} s_\alpha s_\beta\, u,_{\gamma\delta}, \tag{34}$$

eqn (31) is transformed to:

$$\int_\Omega [u^* \Lambda^c(u) - u\Lambda^c(u^*)]\mathrm{d}\Omega + I_b^c(u,u^*) + J^c(u,u^*) = 0 \tag{35}$$

where ϕ is the curvature of the boundary contour Γ and

$$I_b^c(u,u^*) = \int_\Gamma [u^* V^c(u) - M_n^c(u)\theta_n(u^*) + M_n^c(u^*)\theta_n(u) - uV^c(u^*)]\mathrm{d}\Gamma \tag{36}$$

$$J^c(u,u^*) = \sum_{j=1}^{K} \left\{ [\![M_{ns}^c(u)]\!]_j \, u_j^* - [\![M_{ns}^c(u^*)]\!]_j \, u_j \right\} \tag{37}$$

where K is the number of corners along a non-smooth boundary Γ. Relative to the local **n**-**s** frame of reference relative to Γ:

$$M_n^c(u) = -C_{\alpha\beta\gamma\delta} n_\alpha n_\beta (n_\gamma n_\delta \frac{\partial u^2}{\partial n^2} + 2n_\gamma s_\delta \frac{\partial u^2}{\partial n \partial s} + s_\gamma s_\delta \frac{\partial u^2}{\partial s^2})$$

$$= -(C_{nnnn} \frac{\partial u^2}{\partial n^2} + 2C_{nnns} \frac{\partial u^2}{\partial n \partial s} + C_{nnss} \frac{\partial u^2}{\partial s^2}) \tag{38}$$

$$M_{ns}^c(u) = -C_{\alpha\beta\gamma\delta} n_\alpha s_\beta (n_\gamma n_\delta \frac{\partial u^2}{\partial n^2} + 2n_\gamma s_\delta \frac{\partial u^2}{\partial n \partial s} + s_\gamma s_\delta \frac{\partial u^2}{\partial s^2})$$

$$= -(C_{nsnn} \frac{\partial u^2}{\partial n^2} + 2C_{nsns} \frac{\partial u^2}{\partial n \partial s} + C_{nsss} \frac{\partial u^2}{\partial s^2}) \tag{39}$$

Eqn (35) is transformed into a pair of boundary integral equations if u^* is replaced by the fundamental solutions u_λ^* ; $\lambda = 1,2$, satisfying

$$C_{\alpha\beta\gamma\delta}\, u^*_{\lambda},_{\alpha\beta\gamma\delta} = \delta_\lambda(\mathbf{x} - \xi) \tag{40}$$

with

$$\delta_1(\mathbf{x} - \xi) = \delta(\mathbf{x} - \xi) \tag{41}$$

$$\delta_2(\mathbf{x} - \xi) = \partial\delta(\mathbf{x} - \xi)/\partial m(\xi) \tag{42}$$

where $\delta(\mathbf{x} - \xi)$ is the delta function representing a unit action at the source point ξ and **m** a unit vector of arbitrary direction emanating from the source point. Substituting u_λ^* in eqn (35) and taking ξ to the boundary so that **m** becomes normal to it, gives

$$\int_\Omega (\Lambda_C u) u_\lambda^* \mathrm{d}\Omega - k u_\lambda(\xi) + I_C^b(u,\, u_\lambda^*) + J_C(u,\, u_\lambda^*) = 0 \tag{43}$$

where $u_1 = u$, $u_2 = \partial u/\partial m$ and $k = 0.5$ along a smooth boundary. Expressions for u_λ^* can be found in earlier, linear BEM analyses of laminated plates [12].

4 Integral equations for the non-linear problem

Integral eqn (43) is applied to both extensional and flexural problems governed, respectively, by incremental eqns (24) and (25):

$$-k\delta F_\lambda(\xi) + I_A^b(\delta F,\ F_\lambda^*) + J_A(\delta F,\ F_\lambda^*) + I_F^d(w_{,\alpha\beta},\ \delta w_{,\alpha\beta}, F_\lambda^*) = 0 \tag{44}$$

$$-k\delta w_\lambda(\xi) + I_D^b(\delta w, w_\lambda^*) + J_D(\delta w, w_\lambda^*) + I_w^d(F_{,\alpha\beta},\ \delta F_{,\alpha\beta},\ w_{,\alpha\beta},\ \delta w_{,\alpha\beta},\ w_\lambda^*) = 0 \tag{45}$$

where

$$I_F^d(w_{,\alpha\beta},\ \delta w_{,\alpha\beta}, F_\lambda^*) = \int_\Omega (-L_{\alpha\beta}\hat{w} + \frac{1}{2}L_{\alpha\beta}\delta w)\delta w_{,\alpha\beta}\ F_\lambda^* \mathrm{d}\Omega$$

$$I_w^d(F_{,\alpha\beta},\ \delta F_{,\alpha\beta},\ w_{,\alpha\beta},\ \delta w_{,\alpha\beta},\ w_\lambda^*) = \int_\Omega [L_{\alpha\beta}\hat{w}\delta F_{,\alpha\beta} + L_{\alpha\beta}(F+\delta F)\delta w_{,\alpha\beta}]w_\lambda^* \mathrm{d}\Omega$$

The domain values of the stress function and deflection increments are obtained by setting $k = \lambda = 1$ in eqns (44) and (45) and recalling that $F_1 = F$, $F_1^* = F^*$, $w_1 = w$, $w_1^* = w^*$, so that

$$\delta F(\xi) = I_A^b(\delta F, F^*) + J_A(\delta F, F^*) + I_F^d(w_{,\alpha\beta},\ \delta w_{,\alpha\beta},\ F^*) \tag{46}$$

$$\delta w(\xi) = I_D^b(\delta w, w^*) + J_D(\delta w, w^*) + I_w^d(F_{,\alpha\beta},\ \delta F_{,\alpha\beta},\ w_{,\alpha\beta},\ \delta w_{,\alpha\beta},\ w^*) \tag{47}$$

Thus, the incremental 2nd order derivatives of the stress function and deflection are obtained from

$$\delta F_{,\gamma\delta}(\xi) = I_A^b(\delta F, F^*{}_{,\gamma\delta}) + J_A(\delta F, F^*{}_{,\gamma\delta}) + I_F^d(w_{,\alpha\beta},\ \delta w_{,\alpha\beta},\ F^*{}_{,\gamma\delta}) \tag{48}$$

$$\delta w_{,\gamma\delta}(\xi) = I_D^b(\delta w, w^*{}_{,\gamma\delta}) + J_D(\delta w, w^*{}_{,\gamma\delta}) + I_w^d(F_{,\alpha\beta},\ \delta F_{,\alpha\beta},\ w_{,\alpha\beta},\ \delta w_{,\alpha\beta},\ w^*{}_{,\gamma\delta}) \tag{49}$$

5 Solution procedure

Boundary element modelling of the plate contour, introduced to integral eqns (44) and (45) as well as (48) and (49) leads to the incremental solution of the problem governing the post-buckling behaviour of a symmetrically laminated plate. The variables δF and $\partial(\delta F)/\partial n$ in boundary integrals $I_A^b(\delta F, F_\lambda^*;\ \lambda = 1,2)$ on the right-hand side of eqns (44) and (48) are functions of the edge traction increments $\delta\tilde{p}_\alpha$ [12]. The distribution of the other two boundary variables $M_n(\delta F)$ and $V_n(\delta F)$ in the same integrals are unknown.

The deflection-related unknown boundary variables in boundary integrals $I_D^b(\delta w, w_\lambda^*;\ \lambda=1,2)$ on the right-hand side of eqns (45) and (49) are consistent with the specified boundary conditions among eqns (27)-(29). The boundary is discretised into quadratic discontinuous elements for evaluating the boundary integrals and the domain is discretised into linear discontinuous cells for the evaluation of domain integrals. The incremental membrane stresses and curvatures at points inside the domain are obtained using eqns (48) and (49).

All the incremental values are assumed zero at the beginning of each step. At the first step and first iteration, an assumed pattern of very small initial imperfection w^i is considered as the total deflection $\hat{w}$ over the domain. The unknown boundary distributions of $M_n(\delta F)$ and $V_n(\delta F)$ are obtained for an increment of edge traction $\delta \tilde{p}_\alpha$ by solving eqns (44) with I_F^d ($w,_{\alpha\beta}$, $\delta w,_{\alpha\beta}$, F_λ^* ; λ = 1,2) set equal to zero. The incremental membrane stresses at domain cell nodes are obtained using eqn (48). The distributions of unknown deflection-related variables on the boundary are obtained by solving eqns (45). The domain integrals I_w^d ($F,_{\alpha\beta}$, $\delta F,_{\alpha\beta}$, $w,_{\alpha\beta}$, $\delta w,_{\alpha\beta}$, w_λ^* ; λ = 1,2) are evaluated using the incremental membrane stresses obtained previously. The incremental deflections and curvatures at domain cell nodes are obtained using eqns (47) and (49), respectively. Replacing the incremental values of membrane stresses and curvatures by the new values, the procedure is repeated in an iterative manner until the specified convergence criteria are satisfied. The adopted convergence criterion for the deflection after j^{th} iterations is:

$$\frac{\sum_{i=1}^{N_d}\left|(\delta w_i)^j - (\delta w_i)^{j-1}\right|}{\sum_{i=1}^{N_d}\left|(\delta w_i)^j\right|} \leq 0.0005 \qquad (50)$$

where N_d is the number of domain cell nodes. Similar criteria for the incremental membrane stresses are also imposed in order to reduce the error accumulated at each step. The final incremental values of the last iteration in the previous step are added to the membrane stresses, deflections and curvatures for the next step. The solution procedure is carried out in a similar manner for further steps. The procedure at $(i+1)^{\text{th}}$ step is summarised as follows:

(1) From step i, $\hat{w}$, $\hat{w},_{\alpha\beta}$ and $N_{\alpha\beta}$ are known.

(2) For step (i+1), $\delta w,_{\alpha\beta}$ are set equal to zero for the 1st iteration, otherwise taken from the previous iteration.

(3) Domain integrals I_F^d ($w,_{\alpha\beta}$, $\delta w,_{\alpha\beta}$, F_λ^* ; λ = 1,2) are evaluated using the curvatures of the i^{th} step and the incremental curvatures from the previous iteration within the $(i+1)^{\text{th}}$ step.

(4) Eqns (44) are solved for the unknown boundary variables and then the incremental membrane stresses are evaluated at domain cell nodes using eqn (48).

(5) Domain integrals I_w^d ($F,_{\alpha\beta}$, $\delta F,_{\alpha\beta}$, $w,_{\alpha\beta}$, $\delta w,_{\alpha\beta}$, w_λ^* ; λ = 1,2), are evaluated using membrane stresses and curvatures from the i^{th} step and the incremental membrane stresses and curvatures from the previous iteration within the $(i+1)^{\text{th}}$ step.

(6) Eqns (45) are solved for unknown boundary variables and then the incremental deflection and curvatures are evaluated at domain cell nodes using eqns (47) and (49), respectively.

(7) Convergence criteria are applied.

(8) If the convergence criteria are satisfied, the procedure moves to the next step, otherwise steps 2 to 6 are repeated.
(9) Before moving to the next step (i+2), the values of $\hat{w}$, $\hat{w},_{\alpha\beta}$ and $N_{\alpha\beta}$ are updated by adding the incremental values from the last iteration within step (i+1).

6 Results

The proposed BEM formulation for the post-buckling analysis of laminated plates was implemented through a suite of C codes. The analysis was applied to a simply-supported square plate with a side length of 1.0 m and a thickness of 10 mm, subjected to uni-axial, uniformly distributed compressive loads along the stiffer direction. Two cases were analysed corresponding to different initial imperfection patterns with maximum values of 0.5 and 5 mm. The flexural rigidities and extensional stiffness of the plate were

$D_{11} = 16692.7$ Nm	$D_{22} = 417.319$ Nm
$D_{66} = 250.0$ Nm	$D_{12} = 417.319$ Nm
$A_{11} = 2.00313\times10^{9}$ N/m	$A_{22} = 5.00782\times10^{7}$ N/m
$A_{12} = 1.25196\times10^{7}$ N/m	$A_{66} = 3.0\times10^{7}$ N/m

The boundary was discretised into 80 quadratic discontinuous elements with 50 linear domain cells over the plate. The results were compared with the corresponding solution for a perfectly flat plate given by Prabhakara and Chia [1], which was based on a double Fourier series for the transverse deflection and a double series for the stress function consisting of appropriate beam functions. The results are in good agreement for the range of post-buckling deflection shown in Figure 1. The present method relies heavily on iterations and its performance depends upon various factors like step size, the degree of non-linearity, the convergence criteria and the rate change of slope of load-deflection curve. Thus, further parametric studies are required for identifying an optimised procedure for the solution. Alternative solution procedures whereby iterations may be avoided or their number significantly reduced are described in the next section.

7 Concluding remarks

A large deformation analysis leading to the prediction of the post-buckling behaviour was developed as an expansion and coupling of the formulations for linear analyses [12]. This problem is of considerable design interest since its output is linked more closely than the critical load to the strength limits of stiffened composite panels. The analysis performed shows that the present procedure is highly dependent on the size of load step, the number of iterations within each step, which is controlled by convergence criteria, and the degree of non-linearity. The size of load step should be reduced near the critical load and within the post-buckling region. Further validation of the proposed procedure is

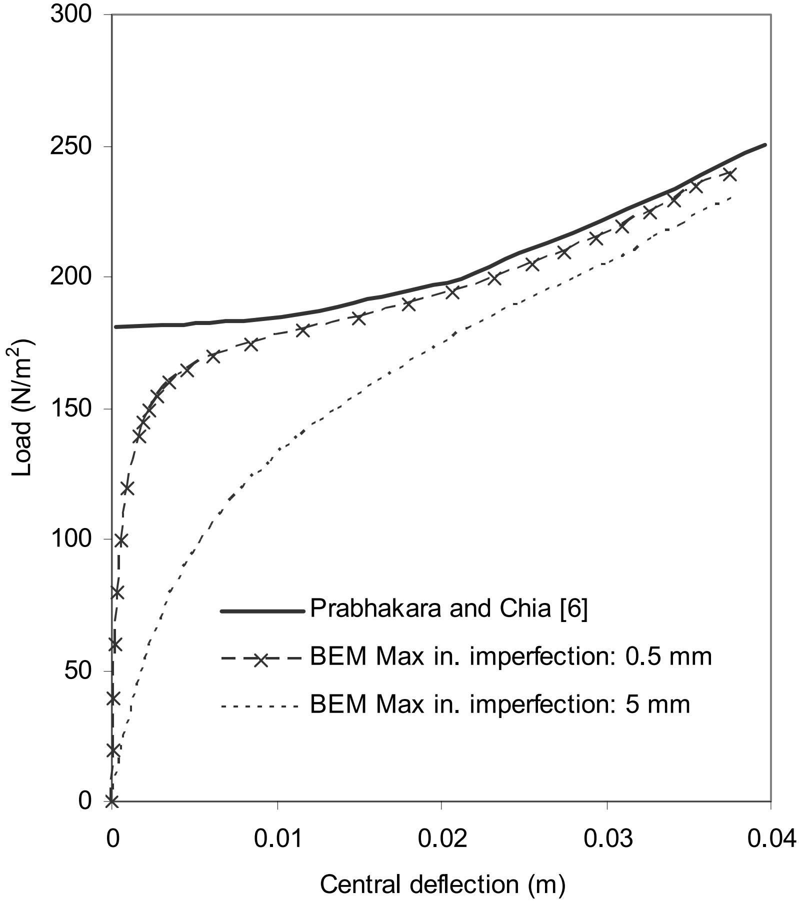

Figure 1: Non-linear load-deflection curves for different initial imperfections for a simply supported plate under uni-axial uniform loading.

required for developing an optimised, efficient analysis in terms of computational cost.

The advantage of the boundary element method relies on the existence of a boundary integral equation that reduces the dimensions of a problem by one thus leading to its more efficient formulation and solution. This basic advantage of BEM is to a certain extent compromised by the presence of irreducible domain integrals in the integral equations governing the plate post-buckling problem. Transformation of such integrals into boundary ones is therefore desirable and could be achieved by adopting suitable approximate representations of the domain unknowns.

The performance of the proposed scheme may be improved by dividing each step into two sub-steps. Within the first of those sub-steps, the solution is carried out without considering the quadratic terms in the unknown increments appearing in the domain integrals of eqns (44) and (45). This will allow the direct solution of eqns (44) and the determination of the incremental membrane stresses over the domain from eqn (48). A consistent system of equations, containing the deflection-related boundary unknowns and the domain curvatures, is also obtained from integral eqns (45) and (49). After determining all incremental quantities, the result is further corrected by including the quadratic terms in the domain integrals and solving the system iteratively within the second sub-step of the current load step.

An alternative scheme would rely on a model for the deflection that would allow the numerical evaluation of the curvatures on which the domain integrals in the post-buckling eqns (44) and (45) depend. This can be achieved by adopting non-linear interpolation models in the form of higher order polynomials or trigonometric functions. The curvatures can be obtained by differentiating these models, which leads to a direct relationship between nodal cell curvatures and deflections. An additional system of equations for the deflections can be obtained from integral eqn (47). By eliminating the membrane stresses and the curvatures over the plate domain, the final system of equations would contain only the deflection-related boundary variables and domain deflections as unknowns. As with the previously described scheme, the solution can be initially carried out neglecting the quadratic terms in the domain integrals. After the determination of all incremental quantities, the result can be further corrected by including these quadratic terms and solving the system iteratively within the current load step. This approach may lead to a reduction and a more efficient use of iterations towards an accurate and stable solution.

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