The radial basis integral equation method for convection-diffusion problems

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Abstract

The Boundary Element Dual Reciprocity Method has been implemented as a meshless approach. The method uses circular sub-domains with overlapping distributed inside the original domain of the problem. Since the source point is always in the centre of the circular sub-domain singular integrals are avoided regardless of the order of the derivative of the original integral equation. Three equations for two-dimensional (2D) potential problems are required at each node. The first equation is the usual BEM integral equation arising from the application of the Green's identities and the remaining equations are the derivatives of the first equation in respect to space coordinates. In the current approach Radial Basis Function interpolation is applied to obtain the values of the field variables and partial derivatives at the boundary of the circular sub-domains. Dual reciprocity method (DRM) has been applied to convert the domain integrals into boundary integrals. The method has been tested on a convection-diffusion problem. The results obtained using the current approach are compared to previously reported results obtained using the Finite Element Method (FEM), and the DRM multi-domain approach (DRM-MD) showing similar level of accuracy.

Keywords: meshless method, integral equations, circular sub-domains, radial basis functions.

1 Introduction

The local boundary integral equation (LBIE) was proposed by Zhu et al. [1, 2]. In the LBIE the domain is sub-divided in a large number of sub-domains in a shape of a circle, with the source point in the centre of the circle. The most often

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used interpolation for field variables were the moving least-squares, though Sellountos and Sequeira [3] used augmented thin plate spline (ATPS) Radial Basis Functions (RBFs) for interpolation of the field variable and gradients over the circular boundaries. The concept of "companion solution" is introduced in order to eliminate the single layer integral from the local boundary integral equation. In this way the potential field is the only unknown in the equations. For source points that are located on the (global) boundary of the given problem integration over the boundary has to be performed.

The present formulation similarly to the LBIE is implemented over circular sub-domains where the source points are placed in the centres of the circles. The work follows the idea of Bui and Popov [4] who proposed using three equations at each source point for 2D problems solved using BEM with overlapping sub-domains. One equation is the original integral equation usually used in the direct formulation BEM, while the other two equations are the derivatives in respect to spatial coordinates of the original equation at the source point. In this work the augmented thin plate spline (ATPS) radial basis functions (RBFs) were used for interpolation of the field variable and gradients over the circular boundaries. This RBF was selected in order to use the same interpolation function for representing the field variables for the approximation in the DRM part of the formulation. Further in this paper it will be referred to the current meshless approach as the radial basis integral equation method (RBIEM).

The LBIE uses the concept of "companion solution" in order to avoid solution for the gradients/normal derivatives inside the problem domain, while the RBIEM solves for the potential and partial derivatives at each node. This enables the RBIEM to be a truly meshless approach since the values of the normal derivatives are obtained everywhere including the source points located on the global boundary of the problem domain. The boundary conditions in the RBIEM are imposed directly at the source points on the global boundary. In the RBIEM there is no need for integration over any part of the global boundary of the problem domain.

The RBIEM always produces a closed system of equations, unlike the DRM-MD and the Boundary-Domain Integral Method (BDIM) which produce over-determined systems of equations.

The RBIEM is especially effective in applications where the partial derivatives in respect to coordinates are required, e.g., the convection-diffusion equation, the Navier-Stokes equation.

Further in the paper the "global boundary" will mean the boundary of the given problem and the "local (circular) boundary" will mean the boundary of the circular sub-domains.

2 The boundary element dual reciprocity method

Let us consider the following equation:

$$\nabla u^2(r) = b\left(r, u(r), \frac{\partial u(r)}{\partial x_i}, \frac{\partial u(r)}{\partial t}\right) \tag{1}$$

where $u(r)$ is a potential field, r is a position vector, x_i is component of r, and t is time. Given a point r *inside a* domain Ω, by applying the Green integral formula equation (1) can be transformed into the following integral form:

$$u(r)+\int_{\Gamma} q^*(r,\xi)u(\xi)d\Gamma_\xi-\int_{\Gamma} u^*(r,\xi)q(\xi)d\Gamma_\xi=\int_{\Omega} u^*(r,\xi)b(\xi)d\Omega_\xi. \tag{2}$$

where $u^*(r,\xi)$ is the fundamental solution of the Laplace problem, $q(\xi)=\partial u(\xi)/\partial n$ and $q^*(r,\xi)=\partial u^*(r,\xi)/\partial n$.

The DRM approximation [5] is introduced to transform the domain integral in (2) in terms of equivalent boundary integrals. The implementation of the DRM with multidomain technique is explained by Natalini and Popov [6]. After application of the DRM, the following integral representation formula is obtained:

$$\begin{aligned} &u(r)+\int_{\Gamma} q^*(r,\xi)u(\xi)d\Gamma_\xi-\int_{\Gamma} u^*(r,\xi)q(\xi)d\Gamma_\xi= \\ &\sum_{k=1}^{J_{bn}+J_{in}}\left\{\alpha_k\left(\hat{u}(r,\eta_k)+\int_{\Gamma} q^*(r,\xi)\hat{u}(\xi,\eta_k)d\Gamma_\xi-\int_{\Gamma} u^*(r,\xi)\hat{q}(\xi,\eta_k)d\Gamma_\xi\right)\right\} \end{aligned} \tag{3}$$

where J_{bn} and J_{in} are the number of boundary nodes and internal nodes on the domain, respectively.

3 The radial basis integral equation method

The proposed formulation solves in each interior node three integral equations in order to obtain the potential u, and the partial derivatives $\partial u/\partial x_j$. Equation (3) is used to find the potential while the equations for derivatives $\partial u/\partial x_j$ are obtained by differentiating (3) in respect to x_j, where x_j are components of r. The derivatives of (3) are given below:

$$\begin{aligned} &\frac{\partial u(r)}{\partial x_j}=-\int_{\Gamma_i}\frac{\partial q^*(r,\xi)}{\partial x_j}u(\xi)d\Gamma_\xi+\int_{\Gamma_i}\frac{\partial u^*(r,\xi)}{\partial x_j}q(\xi)d\Gamma_\xi+ \\ &\sum_{k=1}^{J_{bn}+J_{in}}\left\{\alpha_k\left(\frac{\partial\hat{u}(r,\eta_k)}{\partial x_j}+\int_{\Gamma_i}\frac{\partial q^*(r,\xi)}{\partial x_j}\hat{u}(\xi,\eta_k)d\Gamma_\xi-\int_{\Gamma_i}\frac{\partial u^*(r,\xi)}{\partial x_j}\hat{q}(\xi,\eta_k)d\Gamma_\xi\right)\right\} \end{aligned} \tag{4}$$

The normal derivative q in (4) can be written as:

$$q=\frac{\partial u}{\partial n}=\nabla u\cdot\vec{n}=\sum_k\frac{\partial u}{\partial x_k}n_k \tag{5}$$

where n_k are components of the unit normal vector.

4 Interpolation for the unknown values at the circular boundary of the sub-domain

In order to perform the integration over the local boundaries of the circular sub-domains, values of the potentials and partial derivatives must be known on the circles. Eight fictitious nodes were introduced on the circular boundaries in order

to define four quadratic elements used in the integration over the circles. The values of the field variables at the eight nodes were determined through interpolation using the values of field variables at neighbouring nodes. The final system of equations solves for potentials and derivatives only at source points at centres of circular sub-domains, and not at the fictitious nodes on the circular boundaries. Only nodes at centres of sub-domains are used in the interpolation for obtaining the values of field variables at fictitious nodes on the circular boundaries. The unknown potential at one of the eight nodes, denoted by ω is approximated by n neighboring nodes x_i by the following formula:

$$u(\omega) = \sum_{i=1}^{n} f(\omega, x_i) \cdot a_i \tag{6}$$

Here f is the Augmented Thin Plate Spline Radial Basis function and a_i are the unknown coefficients.

The partial derivatives at ω are interpolated in a similar way:

$$\frac{\partial u(\omega)}{\partial x_l} = F(\omega, x_i) F_0^{-1} q_{0l} \tag{7}$$

where $F_0 = f_{ji} = f(x_j, x_i)$, $q_{0l} = [q_l(x_1), q_l(x_2), \ldots, q_l(x_n)]^T$ and $q_l = \partial u / \partial x_l$.

5 Solution on the boundary of the domain

Unlike the LBIE, the RBIEM does not require integration over any part of the global boundary. When the source point is on the global boundary, the part of the local boundary Γ_i of the sub-domain Ω_i containing the source point would partially be located outside the problem domain Ω, see Figure 1. For the nodes on Γ_i which are outside Ω and Γ, extrapolation for the potential and the partial derivatives is required, as can be seen in Figure 1, in order to be able to solve (3) and (4) at x_ξ. The values extrapolated at ω for parts of circles outside the problem domain are required in the solution procedure, but do not have a physical meaning. However, these values are not presented in the final solution and therefore do not affect the validity of the approach.

The boundary conditions (BC) are imposed at the nodes, e.g. x_ξ, located at the boundary Γ. Therefore, it is necessary to place some of the nodes on Γ in order to define the geometry of the problem and to be able to impose the BCs. If Dirichlet BCs are imposed on the part of the boundary where x_ξ is located, the following equation would be applied at x_ξ

$$u(x_\xi) = U_0 \tag{8}$$

which would reduce the number of equations at x_ξ to two. If Neuman BCs are given on the part of the boundary where x_ξ is located, one of the partial derivatives would be eliminated by using (5) and only two equations would remain at x_ξ.

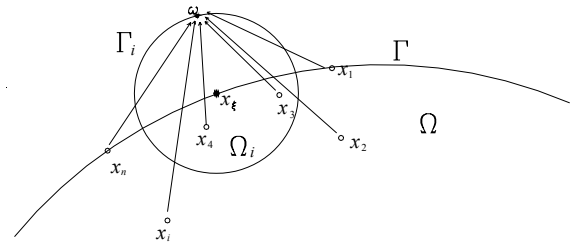

Figure 1: Extrapolation of field variables at a point on Γ_i of sub-domain Ω_i located outside the problem domain Ω.

6 Solution procedures

The RBIEM generates one circular sub-domain for each of the nodes located inside the domain Ω or on the boundary Γ. In Figure 2 several such sub-domains are shown. The eight nodes on the boundary of the circles are introduced as nodes where the potential and derivatives are evaluated using interpolation (6). The values at the eight nodes are than used to calculate the integrals in (3) and (4) at node *i*.

Since the current formulation employs the DRM approach, an approximation procedure is used to represent the non-homogeneous term in (1) as is shown in (4). The DRM approximation employs a number of nodes located around node *i*. It is possible to use different set of nodes for the approximation of the field variables on the boundary of the sub-domain to the ones used for the DRM approximation. In this case, for each sub-domain, the same set of nodes was used for interpolation of the field variables over the circular local boundaries; this means the same set was used for each of the eight nodes on the circle, and for the DRM approximation. This significantly simplifies the search for the neighbouring nodes and saves CPU time.

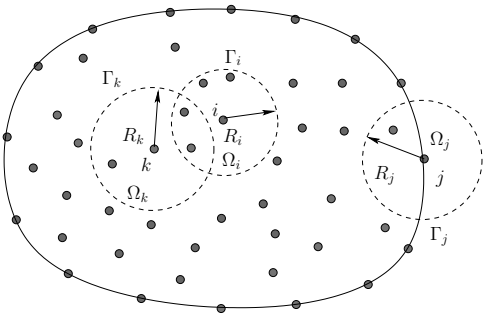

Figure 2: Distribution of some of the circular sub-domains in the problem domain.

7 Numerical examples - convection-diffusion equation

The numerical example used to test the developed approach is a Convection-Diffusion equation with variable velocity field and reaction term [7]:

$$D\frac{d^2u}{dx^2} - V_x\frac{du}{dx} - ku = 0 \tag{9}$$

A rectangular domain with length L and width W is considered. The following BCs were applied:

$$u(0,y) = U_0;\ u(L,y) = U_1;\quad \frac{\partial u}{\partial y}|_{y=W/2} = \frac{\partial u}{\partial y}|_{y=-W/2} = 0 \tag{10}$$

The velocity field is defined as:

$$V_x = \ln\frac{U_1}{U_0} + k\left(x - \frac{1}{2}\right). \tag{11}$$

The analytical solution of the above problem for L = 1m and D =1 m^2s^{-1} is given by:

$$u(x) = U_0 \exp\left\{\frac{k}{2}x^2 + \left(\ln\frac{U_1}{U_0} - \frac{k}{2}\right)x\right\} \tag{12}$$

Two values for k in (11) are used; k = 10 and k = 40. The maximum Peclet number for the examples is Pe_{max} = 8.4, for k = 10, and Pe_{max} = 23.4 for k = 40.

The accuracy of the RBIEM is studied by comparing the numerical results with the analytical solution (12) and with the numerical results obtained from the DRM-MD [7] and the FEM.

In the following examples the distribution of the nodes in the domain was the same irrespective of the approach applied. The distribution of the 185 nodes used and the boundaries of the sub-domains used in the RBIEM are shown in Figure 3. The nodes are located at the centres of circles. In the examples 15 neighbouring nodes are used in the DRM approximation and to approximate the value of nodes on the local circular boundaries.

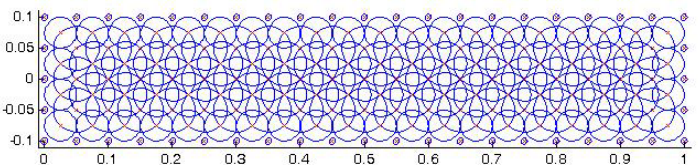

Figure 3: Distribution of the 185 nodes and the size of the circular sub-domains used.

In Table 1 the value of potentials for the case of k = 40 obtained by using the RBIEM are compared with the analytical results and the results obtained using the DRM-MD [7], and the FEM. It can be seen that the RBIEM produces more accurate results than the other two approaches. The only part where RBIEM did not perform better was the first part of the domain, x = 0.02, which is close to the

boundary. It could be the influence of the boundary conditions or the very high gradients in this part of the domain. However, even in this part the error for RBIEM is less than 5.5% and it is the only approach of the presented ones that did not have error higher than 10% within the domain.

Table 1: Potential values at internal points obtained using the RBIEM and compared with the analytical solution, DRM-MD [7], and Galerkin FEM for a mesh with 185 nodes (case k=40).

x	Analytical solution	RBIEM	DRM-MD	FEM (Galerkin)
0.00	300.00000	300.00000	300.00000	300.00000
0.02	189.38055	179.04666	186.34900	187.68159
0.04	121.47817	118.35698	117.97600	119.41530
0.07	64.30894	63.18108	60.98710	61.66200
0.10	35.29222	35.16106	32.87830	33.14000
0.14	16.77039	17.05505	15.19730	15.10380
0.19	7.23952	7.47060	6.38346	6.12290
0.25	3.01465	3.14630	2.60870	2.36960
0.32	1.30135	1.34937	1.12280	0.96120
0.40	0.63337	0.66253	0.55476	0.45580
0.56	0.32339	0.34728	0.29582	0.25190
0.75	0.55040	0.59498	0.52845	0.48900
0.85	1.30052	1.32975	1.29153	1.21380
0.93	3.45102	3.28041	3.46916	3.33370
1.00	10.00000	10.00000	10.00000	10.00000

The results for the derivative of the potential at internal points for the case k=40 obtained by using the RBIEM and DRM-MD are shown in Table 2. It can be seen that RBIEM follows accurately the analytical solution and shows higher errors only at the boundary x=0, where the values are the highest.

7.1 The convergence of the meshless method

To test the convergence different number of nodes were used for the same problem given above. Five different distributions of nodes were used with 95 nodes, 185 nodes, 689 nodes, 1513 nodes and 2657 nodes distributed inside the domain and on the boundary.

The convergence of the proposed method is tested by examining the maximum error and average error of the numerical results compared with the analytical results. The maximum error is defined as

Table 2: Results for the derivative of the potential at internal points obtained using the RBIEM and compared with the analytical solution and DRM-MD for $k = 40$ and mesh with 185 nodes.

x	Analytical	RBIEM	DRM-MD -NOSD
0.00	-7020.35921	-6177.92330	-7161.47021
0.10	-684.71146	-643.60917	-639.22498
0.25	-40.39990	-38.82190	-33.38770
0.40	-4.68772	-4.52786	-3.80930
0.56	-0.32378	-0.19140	-0.23255
0.63	0.59819	0.77870	0.61766
0.75	3.63196	4.08681	3.62576
0.80	6.92022	7.63943	6.89101
0.85	13.78396	14.96371	13.81320
0.93	47.61990	50.71610	46.83430
0.97	95.28866	99.77956	93.10570
1.00	165.98803	165.13214	159.87601

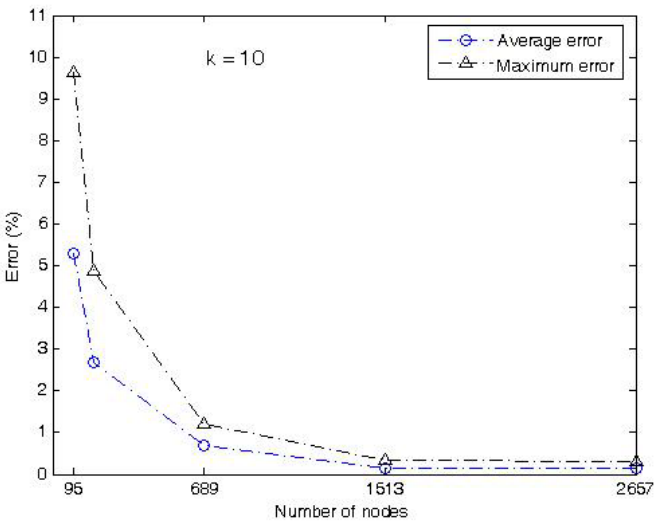

Figure 4: The convergence of the RBIEM (case k=10).

$$e_{\max} = \frac{\max\left|u_i^n - u_i\right|}{u_i} 100\% \tag{13}$$

The average error is defined as

$$e_{average} = \frac{1}{N}\sum_{i=1}^{N}\frac{\left|u_i^n - u_i\right|}{u_i} 100\% \tag{14}$$

where N is number of nodes, u_i^n is the potential at node i obtained by numerical method and u_i is the analytical solution for potential at node i.

In the tests of convergence the values for k = 10 was considered. Figure 4 shows the convergence of the proposed method. It is clear from the figure that the maximum error reduces from 9.6% when 95 nodes are used to 0.32% for the case when 2657 nodes are used. The average error also reduces from 5.27% for the case with 95 nodes to 0.13% for the case with 2657 nodes. It is seen that both maximum error and average error decrease continuously as the number of nodes increase from 95 nodes to 2657 nodes.

8 Conclusions

A meshless method based on the BEM and combined with the sub-domain approach has been proposed for solving the convection-diffusion equation. The approach uses circular sub-domains and places source points in the centres of the sub-domains. Three equations in 2D are solved at each node, where one equation is used for solving the potential and the remaining equations are used for solving the partial derivatives. Radial basis function interpolation is applied in order to obtain the values of the field variable and normal derivatives on the boundary of the circular sub-domains. DRM has been applied to convert the domain integrals into integrals over the boundary of the sub-domains. The current approach does not require any integration over the boundary of the computational domain and the application of the boundary conditions is straightforward. The only evaluation of the integrals is performed over the circular boundaries of the sub-domains. The accuracy of the method has been compared with the accuracy of the DRM-MD with overlapping sub-domains. In all cases the current approach has shown comparable to higher accuracy than the formulations which used a mesh, in this case DRM-MD and the FEM. The method shows good convergence for the tested convection-diffusion problem. Though the method has been applied to 2D problems, extension of the approach to 3D problems is straightforward.

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