Inmost singularities and appropriate quadrature rules in the boundary element method

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Abstract

The solution of singular integral equations (SIEs), taking into consideration the particular behaviour of its regular kernel and its right hand side function, is investigated in this paper. In particular, the problems appearing in the solution of singular integral equations in the boundary element method are verified. It is shown that the behaviour of singular integral equations does not depend only on the behaviour of the regular kernel but on the behaviour of the unknown function.
Keywords: singular integral equation, singularities, nearby poles, elasticity, quadrature formula, numerical integration, boundary element method.

1 Introduction

Many problems in the boundary element method can be reduced to a singular integral equation or to systems of singular integral equations [1]. For the solution of these equations where the known functions are Hölder continuous, approximate solutions through numerical techniques preserving the correct nature of singularities of the unknown function, have been developed [1–10].

Problems in the numerical solution of singular integral equations appear because of the Cauchy singularities, either from the unknown function or from of the poles of the known functions. In addition the regular Kernels of the singular integral equation may have either nearby poles, or branch points, or jump discontinuities.

WIT Transactions on Modelling and Simulation, Vol 44,
www.witpress.com, ISSN 1743-355X (on-line)
doi:10.2495/BE070141

These problems are created in the case of bodies with very narrow boundaries, in the case of the interaction of boundaries (e.g. for a crack approaching a boundary or in the case of parallel cracks when the distance between them becomes very small), and in the case of loading discontinuities. Especially, problems in the solution may appear in one of the following cases:

(i) Singular loadings which creates singularities at the unknown function.
(ii) Collinear or parallel cracks, where the distance between them is very small. There is a singular behaviour because both of them and also a nearby singularity because of the distance. In the case that one of the cracks is eliminated, its influence creates nearby singularities in the regular kernel of the integral equations and it is also influenced the unknown function.
(iii) In the case of a very narrow body. The integral equation created in one of the boundaries, is influenced by the collocation points lying on the second one and vice versa. Thus, a nearby singularity is created inside the kernel of the second one. In the case that one of the boundaries is eliminated, the unknown function is influenced and also a nearby singularity is created at the regular kernel.

In order to confront the above problems modified quadrature rules have to be used. The purpose of this paper is to show and to interpret the above problems, that appear in the solution of singular integral equation in the BEM, and to propose a quadrature rule that may solve successfully the singular integral equations.

2 The influence of poles of the regular kernel

The behavior in the solution of singular integral equations in terms of their regular kernel may appear in the solution of systems of singular integral equations when an integral equation is eliminated and the problem is reduced to a system with modified kernels and/or in the case when there is interaction of boundaries and a complex pole appears in the regular kernel.

A simple example of the above case is the antiplane shear crack under constant loading equal to 1 and approaching perpendicularly either the interface of a bimaterial plane, or the straight boundary of an elastic half-plane. If ε and a are the distances of the transverse-crack tips from the longitudinal straight boundary, the problem is governed by a system of SIEs, one along the crack and the other along the interface. Eliminating the integral along the infinite interface the problem is reduced to the following Cauchy-type singular integral equation:

$$\begin{gathered}\frac{1}{\pi}\int_{\varepsilon}^{\alpha}(t-\varepsilon)^{-1/2}(\alpha-t)^{-1/2}\left[\frac{1}{t-x}+H_1(x,t)\right]q(t)dt=1,\\ H_1(x,t)=\lambda/(t+x);\quad \varepsilon<x<\alpha,\quad \lambda=1,\end{gathered}\tag{1}$$

together with the condition:

$$\int_{\varepsilon}^{\alpha} (t-\varepsilon)^{-1/2} (\alpha - t)^{-1/2} q(t) dt = 0. \tag{2}$$

It is observed the influence of the boundary in the kernel, and a **nearby** singularity appeared in (1) and (2). The dimensionless stress intensity factors at the crack tips $E(t=\varepsilon)$ and $A(t=a)$, are given by

$$K_E = -q(\varepsilon)\left(\frac{a-\varepsilon}{2}\right)^{-1/2}, \quad K_A = -q(a)\left(\frac{a-\varepsilon}{2}\right)^{-1/2}. \tag{3}$$

The stress intensity factors may also be obtained [3] from the following closed-form expressions:

$$K_E = \frac{1}{a-\varepsilon}\left(\frac{2}{a+\varepsilon}\right)^{1/2}\left[a^2 \frac{E(k)}{K(k)} - \varepsilon^2\right]\varepsilon^{-1/2} \tag{4}$$

$$K_A = \frac{1}{a-\varepsilon}\left(\frac{2a^3}{a+\varepsilon}\right)^{1/2}\left[1 - \frac{E(k)}{K(k)}\right], \qquad k = \left(1 - \frac{\varepsilon^2}{a^2}\right)^{1/2}, \tag{5}$$

where $E(k)$ and $K(k)$ elliptic integrals.

As the distance ε tends to zero, K_E tends to infinity like $\varepsilon^{-1/2}/\ell og\varepsilon$, while K_A tends to the value of the stress intensity factor of an equivalent edge crack. For the same problem the asymptotic expressions for the stress intensity factors at $E(t=\varepsilon)$ and $A(t=a=1)$, are given by [4]:

$$K_E = (1-\lambda)^{1/2}\left[\varepsilon^{1/2-s}(1-s)2^{s-1/2} - \varepsilon^{1/2+s}(1+s)2^{-2s-1/2} + O\left(\varepsilon^{2.5-3s}\right)\right],$$
$$K_A = \sqrt{2}(1-\lambda)^{1/2}\left[8 - (1-s)\varepsilon^{2-2s}2^{4s-3} + O\left(\varepsilon^{4-4s}\right)\right], \tag{6}$$

where the quantity s is given in terms of the shear modulus μ_1 and μ_2 by:

$$\cos \pi s = \lambda, \quad 0 < s < 1. \tag{7}$$

Another interesting problem is that of a crack approaching the interface of a bimaterial plane under plane stress or plane strain conditions. Eliminating the singular integral equation along the interface, it is obtained [5]

$$\frac{1}{\pi}\int_{\varepsilon}^{a}(t-\varepsilon)^{-1/2}(a-t)^{-1/2}\left\{\frac{1}{t-x}+H_2(x,t)\right\}q(t)dt=1\,, \tag{8}$$

with

$$H_2(x,t)=\frac{1}{\alpha}\left\{\frac{\gamma+\beta}{t+x}-3\beta\frac{x}{(t+x)^2}+2\beta\frac{x^2}{(t+x)^3}\right\}, \tag{9}$$

where

$$\begin{aligned}\alpha&=(m+\kappa_2)(1+m\kappa_1), \quad \beta=-4(m+\kappa_2)(1-m)\\ \gamma&=(1-m)(m+\kappa_2)+\alpha+\beta/4\,, \quad m=\mu_2/\mu_1\,,\end{aligned} \tag{10}$$

$$\kappa_i=\begin{cases}(3-\nu_i)/(1+\nu_i) & \text{for plane stress}\\ (3\text{-}4\nu_{\mathrm{i}}) & \text{for plane strain}\end{cases} \tag{11}$$

In relations (8) and (9), κ_i $(i=1,2)$ are Muskhelishvili's elastic constants, expressed in terms of Poisson's ratio $\nu_i\,(i=1,2)$ of the corresponding half plane. It is also observed the influence of the eliminating boundary and a *nearby* singularity is appeared in (8). The asymptotic expression of the stress intensity factor $K_E\,(t=\varepsilon)$ obtained by Atkinson [4], is given by:

$$K_E\sim 2^{1/2}\varepsilon^{1/2}\sum_{k=1}^{N}A_k\varepsilon^{-s_k}\ , \quad s_k<1\,, \tag{12}$$

where A_k are constants independent of ε, and s_k are the first N-real roots of the equation:

$$2\alpha\cos\pi s+\beta(s-1)^2+\gamma=0\,. \tag{13}$$

It must be noted that the above equation gives also the eigenvalues of an infinite crack going perpendicularly to an interface as before. In the case where the material 2 is the air $(\mu_2=0)$, the asymptotic expressions of K_E and K_A take the form:

$$K_E\sim\widetilde{K}_A\varepsilon^{-1/2}/\ell og\varepsilon\,, \quad K_A=\widetilde{K}_A(1+1/\ell og\varepsilon)\,, \tag{14}$$

where $\tilde{K}_A$ being the stress intensity factor of an edge crack in a half plane subjected to the same loading.

As a particular example let us consider the case of two collinear cracks $(-a,-\varepsilon)$ and $(0,b)$, i.e. the distance between the nearest tips of the collinear cracks is ε. If we represent by different symbols, ω_1 and ω_2 the quantity $(\Phi^+ - \Phi^-)$, (i.e. the distributions of dislocations) in either crack, it is obtained the following system of singular integral equations:

$$\begin{aligned}
&\frac{1}{\pi}\int_{-a}^{-\varepsilon}\frac{\omega_1(x)}{x-x_1}dx+\frac{1}{\pi}\int_0^b\frac{\omega_2(x)}{x-x_1}dx=p_1, \quad -a<x_1<-\varepsilon\,,\\
&\frac{1}{\pi}\int_{-a}^{-\varepsilon}\frac{\omega_1(x)}{x-x_2}dx+\frac{1}{\pi}\int_0^b\frac{\omega_2(x)}{x-x_2}dx=p_2, \quad 0<x_2<b\,,
\end{aligned} \tag{15}$$

where for the sake of simplicity we have assumed that uniformly distributed loads $p_i(i=1,2)$ are applied to either crack. Furthermore, the following equations, ensuring the single-valuedness of ω_1 and ω_2, must be taken into account:

$$\int_{-a}^{-\varepsilon}\omega_1(x)dx=0\,, \quad \int_0^b\omega_2(x)dx=0\,. \tag{16}$$

Solving the first integral of the first of equations (15) with respect to ω_1 and substituting it into the second of the second of equations (15), we obtain, after some algebra, the following singular integral equation:

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^b\frac{\omega_2(x)}{x-x_2}dx+\frac{1}{2\pi}\int_0^b\left[\frac{(x+a)(x+\varepsilon)}{(x_2+a)(x_2+\varepsilon)}-1\right]^{1/2}\frac{\omega_2(x)}{x-x_2}dx\\
&=p_2-p_1\left\{\frac{2x_2+a+\varepsilon}{\sqrt{(x_2+a)(x_2+\varepsilon)}}-1\right\}, \quad 0\le x_2<b\,.
\end{aligned} \tag{17}$$

This integral equation should be supplemented by the second of equations (16). It is worthwhile noting that the kernel of the second integral in equation (17) is a regular one, possessing a nearby pole at $x_2=-\varepsilon$. Consequently, the equation may be solved by the conventional method, i.e. by taking $\omega_2(x)$ in the following form:

$$\omega_2(x)=x^{-1/2}(x-b)^{-1/2}q(x), \quad 0\le x\le b, \tag{18}$$

where $q(x)$ is a regular function. Such a procedure may give reliable results if ε is sufficiently large, but if ε tends to zero the method fails to yield reliable results. This is due to the fact that the functions $\omega_1(x)$ and $\omega_2(x)$ are equal to the difference of limiting values $\left(\Phi^+ - \Phi^-\right)$ of Muskhelishvili's complex potential $\Phi(z)$, which for this particular problem is of the form:

$$\begin{aligned}
&\Phi(z) = \frac{1}{2\pi i} X(z)\left[p_1 \int_{-a}^{-\varepsilon} \frac{X(t)}{t-z} dt + p_2 \int_0^b \frac{X(t)}{t-z} dt + P_2(z)\right] \\
&X(z) = (z+a)^{-1/2}(z+\varepsilon)^{-1/2} z^{-1/2}(z-b)^{-1/2}, \\
&P_2(z) = c_1 z^2 + c_2(z) + c_3 ,
\end{aligned} \tag{19}$$

where the constants c_1, c_2, c_3 could be determined by the conditions of single-valuedness (16). Consequently, the function $q(x)$ must be written, as follows:

$$q(x) = (x+\varepsilon)^{-1/2}(x+a)^{-1/2} q_1(x), \quad 0 \le x \le b, \tag{20}$$

with $q_1(x)$, a fully regular function.

It is obvious from the last relation, that $q(x)$, for small values of ε, takes significant values in the vicinity of $x = 0$. Especially, the stress intensity factor $K(0)$ corresponding to the tip with $x = 0$ which is proportional to $q(0)$, has a factor $\varepsilon^{-1/2}$.

3 Applications

As application consider the singular integral equation

$$⨍_{-1}^{1} \frac{g(t)}{x-t} dt + \int_{-1}^{1} \frac{g(t)}{(t+ic)} dt = \frac{\pi}{\sqrt{c}\sqrt{1+c}}; \quad g(t) = \left(1-t^2\right)^{-1/2} q(t) \tag{21}$$

with

$$\int_{-1}^{1} g(t) dt = \pi \tag{22}$$

The Gauss-Chebyshev integration rule of the first kind [6] does not converge for $c << 1$. In order to have a good convergence, the proposed interpolatory formulas [10] in the case of Gauss-Chebyshev of the second kind, is applied to the regular term

$$\int_{-1}^{1} \frac{g(t)}{t^2+c}dt \cong \sum_{k=1}^{n} \frac{q(t_k)}{t_k^2+c}\pi \left\{ \frac{1}{n} + \frac{1}{T_n'(t_k)} \frac{\left(\sqrt{c}-\sqrt{c+1}\right)^n}{\sqrt{c+1}} \right. \tag{23}$$

$$\left. \times\left(\frac{t_k}{\sqrt{c}} \sin\frac{(n-1)\pi}{2} + \cos\frac{(n-1)\pi}{2} \right) \right\}; \quad g(t) = \left(1-t^2\right)^{-1/2} q(t)$$

Substituting (23) into (21), we have finally

$$\sum_{k=1}^{n} \frac{\pi}{n} \frac{q(t_k)}{x-t_k} + \sum_{k=1}^{n} \frac{q(t_k)}{t_k^{\ 2}+c}\pi \left\{ \frac{1}{n} + \frac{1}{T_n'(t_k)} \frac{\left(\sqrt{c}-\sqrt{c+1}\right)^n}{\sqrt{c+1}} \right.$$

$$\left. \times\left(\frac{t_k}{\sqrt{c}} \sin\frac{(n-1)\pi}{2} + \cos\frac{(n-1)\pi}{2} \right) \right\} = -\ell n\left|\frac{1-x}{1+x}\right| + \frac{2}{\sqrt{c}}\tan^{-1}\frac{1}{\sqrt{c}}; \tag{24}$$

$$t_k = \cos\frac{2k-1}{2n}\pi, \quad \forall x : U_{n-1}(x) = 0,$$

where $T_n(t)$ and $U_{n-1}(t)$ are the Chebyshev polynomials of the first and second kind [6], respectively.

Taking into consideration (24) and (22) a linear system results, whose solution gives the exact value $q(t)(\equiv 1)$ with machine accuracy for $n \geq 2$.

4 Conclusions

From the previous study it is deduced that, in many problems frequently encountered in the Boundary Element Method, the solution g of the singular integral equation is influenced by other (and perhaps weaker) singularities than the already known singularities, existing at the ends of the integration interval.

A difficult problem that may appear in the solution of singular integral equations arises from the singularities that the regular kernel may possess. The poles of the regular kernel are due to the interaction of the boundaries of a body with a very small distance between them. The above problem continues to exist when a boundary is eliminated (see relation (9)).

Problems where the singular integral equations have regular kernels with poles very close to the integration interval appear in the case of a crack parallel and near to a boundary, in the case of the antiplane shear crack, etc. (Section 2).

In the case that the regular kernel has poles very close to the integration interval $(c << 1)$, the classical Gauss integration rule is impossible to approximate the correct result for a few integration points (Section 3). It is observed that increasing the number of integration points the results for the error of the regular kernel do not improve. This is due to the resulted functional equation which does not represent with a "tolerant" precision the singular integral equation. The convergence is succeeded if only the modified weight quadrature rule which introduces modified weights, is used [10]. The proposed quadrature may be applied for any degree of polynomial because it has been proved [10] that it converges uniformly to the exact value of the integral with the nearby singularity.

References

[1] Brebia, C.A., Telles, J.C.F. and Wrobel, L.S., *Boundary Element Techniques*, Springer-Verlag: Berlin and New York, 1984.

[2] Erdogan, F. and Cook, T.S., Antiplane shear crack terminating at and going through a bimaterial interface. *International Journal of Fracture*, 10, pp. 227-240, 1974.

[3] Tada, H., Paris, P.C. and Irwin, G.R., *The Stress Analysis of Cracks Handbook*, 10.1. Del Research Corp., Hellertown, Pennsylvania, 1973.

[4] Atkinson, C., On the stress intensity factors associated with cracks interacting with an interface between two elastic media. *Int. J. Engineering Science*, 13, pp.489-504, 1975.

[5] Cook, T.S. and Erdogan, F., Stresses in bonded materials with a crack perpendicular to the interface. *Int. J. Engineering Science*, 10, pp.667-697, 1972.

[6] Stroud, A. and Secret, D., *Gaussian quadrature formulas*, Prentice-Hall, Englewood Cliffs, N.J, 1966.

[7] Tsamasphyros, G. and Theocaris, P.S., Equivalence and convergence of direct and indirect methods for the numerical solution of singular integral equations, *Computing*, 31, pp.71-80, 1985.

[8] Theotokoglou, E.E. and Tsamasphyros, G., An integral equation solution for cracked half-planes bonded together and containing debondings along their interface, *International Journal of Fracture*, 55(1), pp. 1-16, 1992.

[9] Lether, F.G., Subtracting out complex singularities in numerical integration. *Math. Comp*. 31(2), pp.223-229, 1997.

[10] Tsamasphyros, G. and Theotokoglou, E. E., A modified formula for integrals with nearby singularities, *International Journal for Numerical Methods in Engineering*, 67, pp. 1082-1093, 2006.